1

ALGORITHMIC PROBLEMS IN AMALGAMS OF FINITE GROUPS

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Abstract. Geometric methods proposed by Stallings [53] for treating finitely generated subgroups of free groups were successfully used to solve a wide collection of decision problems for free groups and their subgroups [4, 25, 37, 38, 43, 48, 56].

It turns out that Stallings' methods can be effectively generalized for the class of amalgams of finite groups [39]. In the present paper we employ subgroup graphs constructed by the generalized Stallings' folding algorithm, presented in [39], to solve various algorithmic problems in amalgams of finite groups.

1. Introduction

Decision (or *algorithmic*) problems is one of the classical subjects of combinatorial group theory, originating in the three fundamental decision problems posed by Dehn [11] in 1911: the *word problem* (which asks to answer whether a word over the group generators represents the identity), the *conjugacy problem* (which asks to answer whether an arbitrary pair of words over the group generators define conjugate elements) and the *isomorphism problem* (which asks to answer whether an arbitrary pair of finite presentations determine isomorphic groups).

Though Dehn solved all three of these problems as restricted to the canonical presentation of fundamental groups of closed 2-manifolds, they are theoretically undecidable (unsolvable) in general [44, 45]. However restrictions to some particular classes of groups may yield surprisingly good results. Remarkable examples include the solvability of the word problem in one-relator groups (Magnus, see II.5.4 in [35]) and in hyperbolic groups (Gromov, see 2.3.B in [17]). The reader is referred to the papers of Miller [44, 45] for a survey of decision problems for groups.

The groups considered in the present paper are amalgams of finite groups. As is well known [2], these groups are hyperbolic. Therefore the word problem in this class of groups is solvable. A natural generalization of the word problem is the *(subgroup) membership problem* (or the *generalized word problem*), which asks to decide whether a word in the generators of the group

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is an element of the given subgroup. An efficient solution of the membership problem in amalgams of finite groups was presented by the author in [39], where graph theoretic methods for treating amalgams of finite groups were developed. Namely, a finitely generated subgroup H of an amalgam $G = G_1 *_A G_2$ of finite groups is canonically represented by a finite labelled graph $\Gamma(H)$. This graph carries all the essential information about the subgroup H itself, which enables one to "read off" a solution of the membership problem in H directly from its subgroup graph $\Gamma(H)$. This yields a quadratic (and sometimes even linear) time solution of the membership problem in amalgams of finite groups.

Such strategy was originally developed by Stallings [53] to treat finitely generated subgroups of free groups. Stallings' approach was topological. He showed that every finitely generated subgroup of a free group is canonically represented by a minimal immersion of a bouquet of circles. Using the graph theoretic language, the results of [53] can be restated as follows. A finitely generated subgroup of a free group is canonically represented by a finite labelled graph which can be constructed algorithmically by a so called process of *Stallings' foldings* (*Stallings' folding algorithm*). Moreover, this algorithm is quadratic in the size of the input [25, 38]. See [55] for a faster implementation of this algorithm.

In [39] Stallings' folding algorithm was generalized to the class of amalgams of finite groups. Along the current paper we refer to this algorithm as the *generalized Stallings' folding algorithm*. Its description is included in the Appendix.

Note that graphs constructed by the Stallings' folding algorithm can be viewed as finite inverse automata as well. This convergence of ideas from the group theory, topology, graph theory, the theory of finite automata and finite semigroups yields reach computational and algorithmic results concerning free groups and their subgroups. In particular, this approach gives polynomial time algorithms to solve the membership problem, the finite index problem, to compute closures of subgroups in various profinite topologies. See [4, 37, 38, 43, 48, 56] for these and other examples of the applications of the Stallings' approach in free groups, and [28, 29, 42, 50] for the applications in some other classes of groups. Note that the Stallings' ideas were recast in a combinatorial graph theoretic way in the remarkable survey paper of Kapovich and Myasnikov [25], where these methods were applied systematically to study the subgroup structure of free groups.

Our objective is to apply the generalized Stallings' methods developed by the author in [39] to solve various decision problems concerning finitely

generated subgroups of amalgams of finite groups algorithmically (that is to find a precise procedure, an *algorithm*), which extends the results of [25].

Our results include polynomial solutions for the following algorithmic problems in amalgams of finite groups, which are known to be unsolvable in general [44, 45]:

- computing subgroup presentations,
- detecting triviality of a given subgroup,
- the freeness problem,
- the finite index problem,
- the separability problem,
- the conjugacy problem,
- the normality,
- the intersection problem,
- the malnormality problem,
- the power problem,
- reading off Kurosh decomposition for finitely generated subgroups of free products of finite groups.

These results are spread out between three papers: [40, 41] and the current one. In [41] free products of finite groups are considered, and an efficient procedure to read off a Kurosh decomposition is presented.

The splitting between [40] and the current paper was done with the following idea in mind. It turn out that some subgroup properties, such as computing of a subgroup presentation and index, as well as detecting of freeness and normality, can be obtained directly by an analysis of the corresponding subgroup graph. Solutions of others require some additional constructions. Thus, for example, intersection properties can be examined via product graphs, and separability needs constructions of a pushout of graphs.

In the current paper algorithmic problems of the first type are presented: the computing of subgroup presentations, the freeness problem and the finite index problem. The separability problem is also included here, because it is closely related with the other problems presented in the current paper. The rest of the algorithmic problems are introduced in [40].

The paper is organized as follows. The Preliminary Section includes the description of the basic notions used along the present paper. Readers familiar with amalgams, normal words in amalgams and labelled graphs can skip it. The next section presents a summary of the results from [39] which are essential for our algorithmic purposes. It describes the nature and the properties of the subgroup graphs constructed by the generalized Stallings' folding algorithm in [39]. The rest of the sections are titled by the names of

various algorithmic problems and present definitions (descriptions) and solutions of the corresponding algorithmic problems. The relevant references to other papers considering similar problems and a rough analysis of the complexity of the presented solutions (algorithms) are provided. In contrast with papers that establish the exploration of the complexity of decision problems as their main goal (for instance, [26, 27, 55]), we do it rapidly (sketchy) viewing in its analysis a way to emphasize the effectiveness of our methods.

Other Methods. There have been a number of papers, where methods, not based on Stallings' foldings, have been presented. One can use these methods to treat finitely generated subgroups of amalgams of finite groups. A topological approach can be found in works of Bogopolskii [5, 6]. For the automata theoretic approach, see papers of Holt and Hurt [23, 24], papers of Cremanns, Kuhn, Madlener and Otto [10, 30], as well as the recent paper of Lohrey and Senizergues [31].

However the methods for treating finitely generated subgroups presented in the above papers were applied to some particular subgroup property. No one of these papers has as its goal a solution of various algorithmic problems, which we consider as our primary aim. Moreover, similarly to the case of free groups (see [25]), our combinatorial approach seems to be the most natural one for this purpose.

2. Acknowledgments

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3. Preliminaries

Amalgams. Let $G = G_1 \ast_A G_2$ be a free product of G_1 and G_2 with amalgamation, customary, an *amalgam* of G_1 and G_2. We assume that the (free) factors are given by the finite group presentations

$$G_1 = gp\langle X_1 | R_1 \rangle, \quad G_2 = gp\langle X_2 | R_2 \rangle \text{ such that } X_1^{\pm} \cap X_2^{\pm} = \emptyset. \tag{1.a}$$

$A = \langle Y \rangle$ is a group such that there exist two monomorphisms

$$\phi_1 : A \to G_1 \text{ and } \phi_2 : A \to G_2. \tag{1.b}$$

Thus G has a finite group presentation

$$G = gp\langle X_1, X_2 | R_1, R_2, \phi_1(a) = \phi_2(a), \ a \in Y \rangle. \tag{1.c}$$

We put $X = X_1 \cup X_2$, $R = R_1 \cup R_2 \cup \{\phi_1(a) = \phi_2(a) \mid a \in Y\}$. Thus $G = gp\langle X|R\rangle$.

As is well known [35, 36, 51], the free factors embed in G. It enables us to identify A with its monomorphic image in each one of the free factors. Sometimes in order to make the context clear we use $\boxed{G_i \cap A}$ [1] to denote the monomorphic image of A in G_i $(i \in \{1, 2\})$.

Elements of $G = gp\langle X|R\rangle$ are equivalence classes of words. However it is customary to blur the distinction between a word u and the equivalence class containing u. We will distinguish between them by using different equality signs: $\boxed{\text{“}\equiv\text{”}}$ for the equality of two words and $\boxed{\text{“}=_G\text{”}}$ to denote the equality of two elements of G, that is the equality of two equivalence classes. Thus in $G = gp\langle x \mid x^4\rangle$, for example, $x \equiv x$ but $x \not\equiv x^{-3}$, while $x =_G x^{-3}$.

Normal Forms. Let $G = G_1 *_A G_2$. A word $g_1 g_2 \cdots g_n \in G$ is *in normal form* (or, simply, it is a *normal word*) if:

(1) $g_i \neq_G 1$ lies in one of the factors, G_1 or G_2,
(2) g_i and g_{i+1} are in different factors,
(3) if $n \neq 1$, then $g_i \notin A$.

We call the sequence $(g_1, g_2, \ldots, g_n)$ a *normal decomposition* of the element $g \in G$, where $g =_G g_1 g_2 \cdots g_n$.

Any $g \in G$ has a representative in a normal form (see, for instance, p.187 in [35]). If $g \equiv g_1 g_2 \cdots g_n$ is in normal form and $n > 1$, then the Normal Form Theorem (IV.2.6 in [35]) implies that $g \neq_G 1$. The number n is unique for a given element g of G and it is called the *syllable length* of g. We denote it $l(g)$. We use $|g|$ to denote the length of g as a word in X^*.

Labelled graphs. Below we follow the notation of [14, 53].

A graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$, and two functions $E(\Gamma) \to E(\Gamma)$ and $E(\Gamma) \to V(\Gamma)$: for each $e \in E$ there is an element $\overline{e} \in E(\Gamma)$ and an element $\iota(e) \in V(\Gamma)$, such that $\overline{\overline{e}} = e$ and $\overline{e} \neq e$.

The elements of $E(\Gamma)$ are called *edges*, and an $e \in E(\Gamma)$ is a *direct edge* of Γ, $\overline{e}$ is the *reverse (inverse) edge* of e.

The elements of $V(\Gamma)$ are called *vertices*, $\iota(e)$ is the *initial vertex* of e, and $\tau(e) = \iota(\overline{e})$ is the *terminal vertex* of e. We call them the *endpoints* of the edge e.

A *path of length* n is a sequence of n edges $p = e_1 \cdots e_n$ such that $v_i = \tau(e_i) = \iota(e_{i+1})$ $(1 \leq i < n)$. We call p a *path from* $v_0 = \iota(e_1)$ *to* $v_n = \tau(e_n)$. The *inverse* of the path p is $\overline{p} = \overline{e_n} \cdots \overline{e_1}$. A path of length 0 is the *empty path*.

[1] Boxes are used for emphasizing purposes only.

We say that the graph Γ is *connected* if $V(\Gamma) \neq \emptyset$ and any two vertices are joined by a path. The path p is *closed* if $\iota(p) = \tau(p)$, and it is *freely reduced* if $e_{i+1} \neq \overline{e_i}$ $(1 \leq i < n)$. Γ is a *tree* if it is a connected graph and every closed freely reduced path in Γ is empty.

A *subgraph* of Γ is a graph C such that $V(C) \subseteq V(\Gamma)$ and $E(C) \subseteq E(\Gamma)$. In this case, by abuse of language, we write $C \subseteq \Gamma$. Similarly, whenever we write $\Gamma_1 \cup \Gamma_2$ or $\Gamma_1 \cap \Gamma_2$, we always mean that the set operations are, in fact, applied to the vertex sets and the edge sets of the corresponding graphs.

A *labelling* of Γ by the set $X^{\pm}$ is a function

$$lab: \ E(\Gamma) \rightarrow X^{\pm}$$

such that for each $e \in E(\Gamma)$, $lab(\overline{e}) \equiv (lab(e))^{-1}$.

The last equality enables one, when representing the labelled graph Γ as a directed diagram, to represent only X-labelled edges, because X^{-1}-labelled edges can be deduced immediately from them.

A graph with a labelling function is called a *labelled (with $X^{\pm}$) graph*. The only graphs considered in the present paper are labelled graphs.

A labelled graph is called *well-labelled* if

$$\iota(e_1) = \iota(e_2), \ lab(e_1) \equiv lab(e_2) \ \Rightarrow \ e_1 = e_2,$$

for each pair of edges $e_1, e_2 \in E(\Gamma)$. See Figure 1.

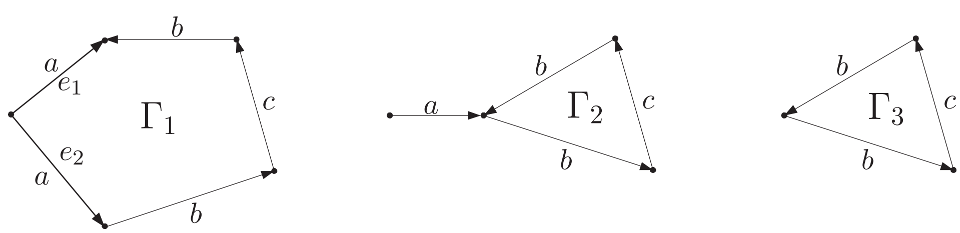

FIGURE 1. The graph Γ_1 is labelled with $\{a, b, c\}^{\pm}$, but it is not well-labelled. The graphs Γ_2 and Γ_3 are well-labelled with $\{a, b, c\}^{\pm}$.

If a finite graph Γ is not well-labelled then a process of iterative identifications of each pair $\{e_1, e_2\}$ of distinct edges with the same initial vertex and the same label to a single edge yields a well-labelled graph. Such identifications are called *foldings*, and the whole process is known as the process of *Stallings' foldings* [4, 25, 37, 38].

Thus the graph Γ_2 on Figure 1 is obtained from the graph Γ_1 by folding the edges e_1 and e_2 to a single edge labelled by a.

Notice that the graph Γ_3 is obtained from the graph Γ_2 by removing the edge labelled by a whose initial vertex has degree 1. Such an edge is called a *hair*, and the above procedure is used to be called *"cutting hairs"*.

The label of a path $p = e_1 e_2 \cdots e_n$ in Γ, where $e_i \in E(\Gamma)$, is the word

$$lab(p) \equiv lab(e_1) \cdots lab(e_n) \in (X^{\pm})^*.$$

Notice that the label of the empty path is the empty word. As usual, we identify the word $lab(p)$ with the corresponding element in $G = gp\langle X|R\rangle$. We say that p is a *normal path* (or p is a path in *normal form*) if $lab(p)$ is a normal word.

If Γ is a well-labelled graph then a path p in Γ is freely reduced if and only if $lab(p)$ is a freely reduced word. Otherwise p can be converted into a freely reduced path p' by iterative removals of the subpaths $e\overline{e}$ (*backtrackings*) ([37, 25]). Thus

$$\iota(p') = \iota(p), \ \tau(p') = \tau(p) \ \text{ and } \ lab(p) =_{FG(X)} lab(p'),$$

where $\boxed{FG(X)}$ is a free group with a free basis X. We say that p' is obtained from p by *free reductions*.

If $v_1, v_2 \in V(\Gamma)$ and p is a path in Γ such that

$$\iota(p) = v_1, \ \tau(p) = v_2 \text{ and } lab(p) \equiv u,$$

then, following the automata theoretic notation, we simply write $\boxed{v_1 \cdot u = v_2}$ to summarize this situation, and say that the word u is *readable* at v_1 in Γ.

A pair $\boxed{(\Gamma, v_0)}$ consisting of the graph Γ and the *basepoint* v_0 (a distinguished vertex of the graph Γ) is called a *pointed graph*.

Following the notation of Gitik ([14]) we denote the set of all closed paths in Γ starting at v_0 by $\boxed{Loop(\Gamma, v_0)}$, and the image of $lab(Loop(\Gamma, v_0))$ in $G = gp\langle X|R\rangle$ by $\boxed{Lab(\Gamma, v_0)}$. More precisely,

$$Loop(\Gamma, v_0) = \{p \mid p \text{ is a path in } \Gamma \text{ with } \iota(p) = \tau(p) = v_0\},$$

$$Lab(\Gamma, v_0) = \{g \in G \mid \exists p \in Loop(\Gamma, v_0) \ : \ lab(p) =_G g\}.$$

It is easy to see that $Lab(\Gamma, v_0)$ is a subgroup of G ([14]). Moreover, $Lab(\Gamma, v) = gLab(\Gamma, u)g^{-1}$, where $g =_G lab(p)$, and p is a path in Γ from v to u ([25]). If $V(\Gamma) = \{v_0\}$ and $E(\Gamma) = \emptyset$ then we assume that $H = \{1\}$.

We say that $H = Lab(\Gamma, v_0)$ is *the subgroup of* G *determined by the graph* (Γ, v_0). Thus any pointed graph labelled by $X^{\pm}$, where X is a generating set of a group G, determines a subgroup of G. This argues the use of the name *subgroup graphs* for such graphs.

Morphisms of Labelled Graphs. Let Γ and Δ be graphs labelled with $X^{\pm}$. The map $\pi : \Gamma \to \Delta$ is called a *morphism of labelled graphs*, if π takes vertices to vertices, edges to edges, preserves labels of direct edges and has the property that

$$\iota(\pi(e)) = \pi(\iota(e)) \text{ and } \tau(\pi(e)) = \pi(\tau(e)), \ \forall e \in E(\Gamma).$$

An injective morphism of labelled graphs is called an *embedding*. If π is an embedding then we say that the graph Γ *embeds* in the graph Δ.

A *morphism of pointed labelled graphs* $\pi : (\Gamma_1, v_1) \to (\Gamma_2, v_2)$ is a morphism of underlying labelled graphs $\pi : \Gamma_1 \to \Gamma_2$ which preserves the basepoint $\pi(v_1) = v_2$. If Γ_2 is well-labelled then there exists at most one such morphism ([25]).

Remark 3.1 ([25])**.** If two pointed well-labelled (with $X^{\pm}$) graphs (Γ_1, v_1) and (Γ_2, v_2) are isomorphic, then there exists a unique isomorphism $\pi : (\Gamma_1, v_1) \to (\Gamma_2, v_2)$. Therefore (Γ_1, v_1) and (Γ_2, v_2) can be identified via π. In this case we sometimes write $(\Gamma_1, v_1) = (\Gamma_2, v_2)$. $\diamond$

The notation $\Gamma_1 = \Gamma_2$ means that there exists an isomorphism between these two graphs. More precisely, one can find $v_i \in V(\Gamma_i)$ $(i \in \{1, 2\})$ such that $(\Gamma_1, v_1) = (\Gamma_2, v_2)$ in the sense of Remark 3.1.

4. Subgroup Graphs

The current section is devoted to the discussion on subgroup graphs constructed by the generalized Stallings' folding algorithm. The main results of [39] concerning these graphs (more precisely, Theorem 7.1, Lemma 8.6, Lemma 8.7, Theorem 8.9 and Corollary 8.11 in [39]), which are essential for the current paper, are summarized in Theorem 4.1 below. All the missing notations are explained along the rest of the present section.

Theorem 4.1. *Let $H = \langle h_1, \cdots, h_k \rangle$ be a finitely generated subgroup of an amalgam of finite groups $G = G_1 *_A G_2$.*

Then there is an algorithm (<u>the generalized Stallings' folding algorithm</u>) which constructs a finite labelled graph $(\Gamma(H), v_0)$ with the following properties:

1. *$Lab(\Gamma(H), v_0) = H$.*
2. *Up to isomorphism, $(\Gamma(H), v_0)$ is a unique <u>reduced precover</u> of G determining H.*
3. *A <u>normal word</u> $g \in G$ is in H if and only if it labels a closed path in $\Gamma(H)$ starting at v_0, that is $v_0 \cdot g = v_0$.*
4. *Let m be the sum of the lengths of words $h_1, \ldots h_n$. Then the algorithm computes $(\Gamma(H), v_0)$ in time $O(m^2)$. Moreover, $|V(\Gamma(H))|$ and $|E(\Gamma(H))|$ are proportional to m.*

Corollary 4.2. *Theorem 4.1 (3) provides a solution of the <u>membership problem</u> for finitely generated subgroups of amalgams of finite groups.*

Throughout the present paper the notation $\boxed{(\Gamma(H), v_0)}$ is used for the finite labelled graph constructed by the generalized Stallings' folding algorithm for a finitely generated subgroup H of an amalgam of finite groups $G = G_1 *_A G_2$.

Definition of Precovers: The notion of *precovers* was defined by Gitik in [14] for subgroup graphs of amalgams. Below we present its definition and list some basic properties. In doing so, we rely on the notation and results obtained in [14].

In [39] some special cases of precovers, *reduced precovers*, were considered. However the properties of *reduced precovers* are irrelevant for the results presented in the current paper. Hence we skip the discussion on them, which can be found in [39].

Let Γ be a graph labelled with $X^{\pm}$, where $X = X_1 \cup X_2$ is the generating set of $G = G_1 *_A G_2$ given by (1.a)-(1.c). We view Γ as a two colored graph: one color for each one of the generating sets X_1 and X_2 of the factors G_1 and G_2, respectively.

The vertex $v \in V(\Gamma)$ is called X_i*-monochromatic* if all the edges of Γ incident with v are labelled with $X_i^{\pm}$, for some $i \in \{1, 2\}$. We denote the set of X_i-monochromatic vertices of Γ by $VM_i(\Gamma)$ and put $VM(\Gamma) = VM_1(\Gamma) \cup VM_2(\Gamma)$.

We say that a vertex $v \in V(\Gamma)$ is *bichromatic* if there exist edges e_1 and e_2 in Γ with

$$\iota(e_1) = \iota(e_2) = v \text{ and } lab(e_i) \in X_i^{\pm},\ i \in \{1, 2\}.$$

The set of bichromatic vertices of Γ is denoted by $VB(\Gamma)$.

A subgraph of Γ is called *monochromatic* if it is labelled only with $X_1^{\pm}$ or only with $X_2^{\pm}$. An X_i*-monochromatic component* of Γ ($i \in \{1, 2\}$) is a maximal connected subgraph of Γ labelled with $X_i^{\pm}$, which contains at least one edge. Thus monochromatic components of Γ are graphs determining subgroups of the factors, G_1 or G_2.

We say that a graph Γ is G*-based* if any path $p \subseteq \Gamma$ with $lab(p) =_G 1$ is closed. Thus if Γ is G-based then, obviously, it is well-labelled with $X^{\pm}$.

Definition 4.3 (Definition of Precover)**.** *A G-based graph Γ is a* precover *of G if each X_i-monochromatic component of Γ is a* cover *of G_i ($i \in \{1, 2\}$).*

Following the terminology of Gitik ([14]), we use the term *"covers of G"* for *relative (coset) Cayley graphs* of G and denote by $\boxed{Cayley(G, S)}$ the

coset Cayley graph of G relative to the subgroup S of G.[2] If $S = \{1\}$, then $Cayley(G,S)$ is the *Cayley graph* of G and the notation $\boxed{Cayley(G)}$ is used.

Note that the use of the term "covers" is adjusted by the well known fact that a geometric realization of a coset Cayley graph of G relative to some $S \leq G$ is a 1-skeleton of a topological cover corresponding to S of the standard 2-complex representing the group G (see [54], pp.162-163).

Convention 4.4. *By the above definition, a precover doesn't have to be a connected graph. However along this paper we restrict our attention only to connected precovers. Thus any time this term is used, we always mean that the corresponding graph is connected unless it is stated otherwise.*

We follow the convention that a graph Γ with $V(\Gamma) = \{v\}$ and $E(\Gamma) = \emptyset$ determining the trivial subgroup (that is $Lab(\Gamma, v) = \{1\}$) is a (an empty) precover of G. $\diamond$

Example 4.5. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3\rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Recall that G is isomorphic to $SL(2,\mathbb{Z})$ under the homomorphism

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; y \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The graphs Γ_1 and Γ_3 on Figure 2 are examples of precovers of G with one monochromatic component and two monochromatic components, respectively.

Though the $\{x\}$-monochromatic component of the graph Γ_2 is a cover of $\mathbb{Z}_4$ and the $\{y\}$-monochromatic component is a cover of $\mathbb{Z}_6$, Γ_2 is not a precover of G, because it is not a G-based graph. Indeed, $v \cdot (x^2 y^{-3}) = u$, while $x^2 y^{-3} =_G 1$.

The graph Γ_4 is not a precover of G because its $\{x\}$-monochromatic components are not covers of $\mathbb{Z}_4$. $\diamond$

A graph Γ is *x-saturated* at $v \in V(\Gamma)$, if there exists $e \in E(\Gamma)$ with $\iota(e) = v$ and $lab(e) = x$ $(x \in X)$. Γ is *$X^{\pm}$-saturated* if it is x-saturated for each $x \in X^{\pm}$ at each $v \in V(\Gamma)$.

Lemma 4.6 (Lemma 1.5 in [14])**.** *Let $G = gp\langle X|R\rangle$ be a group and let (Γ, v_0) be a graph well-labelled with $X^{\pm}$. Denote $Lab(\Gamma, v_0) = S$. Then*

- *Γ is G-based if and only if it can be embedded in $(Cayley(G,S), S \cdot 1)$,*
- *Γ is G-based and $X^{\pm}$-saturated if and only if it is isomorphic to $(Cayley(G,S), S \cdot 1)$.*[3]

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[2]Whenever the notation $Cayley(G,S)$ is used, it always means that S is a subgroup of the group G and the presentation of G is fixed and clear from the context.

[3]We write $S \cdot 1$ instead of the usual $S1 = S$ to distinguish this vertex of $Cayley(G,S)$ as the basepoint of the graph.

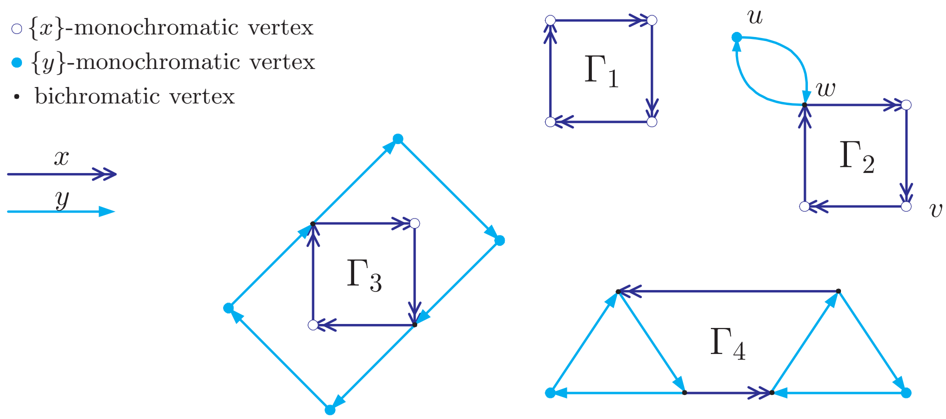

FIGURE 2.

Corollary 4.7. *If Γ is a precover of G with $Lab(\Gamma, v_0) = H \leq G$ then Γ is a subgraph of $Cayley(G, H)$.*

Thus a precover of G can be viewed as a part of the corresponding cover of G, which explains the use of the term "precovers".

Remark 4.8 ([39])**.** Let $\phi : \Gamma \to \Delta$ be a morphism of labelled graphs. If Γ is a precover of G, then $\phi(\Gamma)$ is a precover of G as well. $\diamond$

Precovers are Compatible: A graph Γ is called *compatible at a bichromatic vertex* v if for any monochromatic path p in Γ such that $\iota(p) = v$ and $lab(p) \in A$ there exists a monochromatic path t of a different color in Γ such that $\iota(t) = v$, $\tau(t) = \tau(p)$ and $lab(t) =_G lab(p)$. We say that Γ is *compatible* if it is compatible at all bichromatic vertices.

Example 4.9. The graphs Γ_1 and Γ_3 on Figure 2 are compatible. The graph Γ_2 does not possess this property because $w \cdot x^2 = v$, while $w \cdot y^3 = u$. Γ_4 is not compatible as well. $\diamond$

Lemma 4.10 (Lemma 2.12 in [14])**.** *If Γ is a compatible graph, then for any path p in Γ there exists a path t in normal form such that $\iota(t) = \iota(p)$, $\tau(t) = \tau(p)$ and $lab(t) =_G lab(p)$.*

Remark 4.11 (Remark 2.11 in [14])**.** Precovers are compatible. $\diamond$

The following can be taken as another definition of precovers.

Lemma 4.12 (Corollary2.13 in [14])**.** *Let Γ be a compatible graph. If all X_i-components of Γ are G_i-based, $i \in \{1, 2\}$, then Γ is G-based. In particular, if each X_i-component of Γ is a cover of G_i, $i \in \{1, 2\}$, and Γ is compatible, then Γ is a precover of G.*

Complexity Issues: As were noted in [39], the complexity of the generalized Stallings' algorithm is quadratic in the size of the input, when we assume that all the information concerning the finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c) (see Section 3) is not a part of the input. We also assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given for "free" as well.

Otherwise, if the group presentations of the free factors G_1 and G_2, as well as the monomorphisms between the amalgamated subgroup A and the free factors are a part of the input (the *uniform version* of the algorithm) then we have to build the groups G_1 and G_2, that is to construct their Cayley graphs and relative Cayley graphs.

Since we assume that the groups G_1 and G_2 are finite, the Todd-Coxeter algorithm and the Knuth Bendix algorithm are suitable [35, 52, 54] for these purposes. Then the complexity of the construction depends on the group presentation of G_1 and G_2 we have: it could be even exponential in the size of the presentation [9]. Therefore the generalized Stallings algorithm, presented in [39], with these additional constructions could take time exponential in the size of the input.

Thus each uniform algorithmic problem for H whose solution involves the construction of the subgroup graph $\Gamma(H)$ may have an exponential complexity in the size of the input.

The primary goal of the complexity analysis introduced along the current paper is to estimate our graph theoretical methods. To this end, we assume that all the algorithms along the present paper have the following "given data".

> **GIVEN:** : Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c).
> We assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given.

5. Computing Subgroup Presentations

Given a presentation of a group G, and a suitable information about a subgroup H of G, the Reidemeister-Schreier method (see 2.2.3 in [36]) enables one to compute a presentation for H.

It's a well known fact (see, for instance, [36], p.90) that if $[G : H] < \infty$ then H is finitely generated when G is finitely generated, and H is finitely presented when G is finitely presented. Such a finite presentation of H can be effectively calculated by an application of the Reidemeister-Schreier method.

However a subgroup can be finitely presented even if its index is infinite. For instance, if the group under the consideration is *coherent*, then all its finitely generated subgroups are finitely presented. Recently, coherence of some classes of groups has been investigated in [42, 46].

Below we introduce a restricted version of the Reidemeister-Schreier method which allows to compute a finite presentation for a finitely generated subgroup H of an amalgam of finite groups $G = G_1 *_A G_2$ given by (1.a)-(1.c). This immediately implies the *coherence* of amalgams of finite groups.

The suitable information about the subgroup which is needed for an application of the method can be read off from its subgroup graph $\Gamma(H)$ constructed by the generalized Stallings' algorithm.

Let (Γ, v_0) be a finite precover of G. Let $H = Lab(\Gamma, v_0)$.

Recall that $H = Lab(\Gamma, v_0)$ is the image of $lab(Loop(\Gamma, v_0)) \subseteq X^*$ in G under the natural morphism $\varphi : X^* \to G$. Note that $\varphi = \varphi_2 \circ \varphi_1$, where

$$\varphi_1 : X^* \to FG(X) \text{ and } \varphi_2 : FG(X) \to G.$$

Let $\widetilde{H} = \varphi_1 \left(lab(Loop(\Gamma, v_0)) \right)$. Thus $H = \varphi_2(\widetilde{H})$. Moreover,

$$H = \widetilde{H}/N = \widetilde{H}/\left(\widetilde{H} \cap N\right),$$

where N is the normal closure of R in $FG(X)$ (see [35, 36]). We put $F = FG(X)$.

Let T be a fixed spanning tree of Γ. For all $v \in V(\Gamma)$, we consider t_v to be the unique freely reduced path in T from the basepoint v_0 to the vertex v.

For each $e \in E(\Gamma)$ we consider $t(e) = t_{\iota(e)} e \overline{t_{\tau(e)}}$. Thus if $e \in E(T)$ then $t(e)$ can be freely reduced to an empty path, that is $lab(t(e)) =_F 1$.

Let E^+ be the set of positively oriented edges of Γ. Let

$$X_H = \{ lab(t(e)) \mid e \in E^+ \setminus E(T) \}, \tag{1}$$

$$Q_v = \{ q \subseteq \Gamma \mid \iota(q) = \tau(q) = v, \; lab(q) \equiv r \in R \},$$

$$R_H = \left\{ lab \left(\phi \left(t_v q \overline{t_v} \right) \right) \; | v \in V(\Gamma), \; t_v \subseteq T, \; q \in Q_v, \right\}, \tag{2}$$

where ϕ is a function from the set of freely reduced paths in Γ into $Loop(\Gamma, v_0)$ defined as follows.

$$\phi(p) = t(e_1)t(e_2) \cdots t(e_n), \text{ where } p = e_1 e_2 \cdots e_n \subseteq \Gamma.$$

Thus the path $\phi(p)$ is closed at v_0 in Γ and

$$lab(\phi(p)) \equiv lab(t(e_1)) lab(t(e_2)) \cdots lab(t(e_n)).$$

Moreover, if the path p is closed at v_0 in Γ then the path $\phi(p)$ is *freely equivalent* to p, that is $\phi(p)$ can be transformed to the path p by a series of free reductions. Thus $lab(\phi(p)) =_F lab(p)$.

The function ϕ induces a partial function ϕ' from $FG(X)$ into $FG(X_H)$ such that $\phi'(w) = lab(\phi(p))$, where p is a path in Γ with $lab(p) \equiv w$. Thus another definition of R_H takes the following form

$$R_H = \left\{ \phi' \left(lab(t_v q \overline{t_v}) \right) \mid v \in V(\Gamma),\ t_v \subseteq T,\ q \in Q_v, \right\}. \tag{3}$$

Remark 5.1. Note that the system of coset representatives $\{lab(t_v) \mid v \in V(\Gamma)\}$ is a subset of the *Schreier transversal* of $\widetilde{H}$ in $FG(X)$ ([53]). $\diamond$

Theorem 5.2. *With the above notation, $H = gp\langle X_H \mid R_H \rangle$.*

Proof. As is well known ([25, 37, 53]), $\widetilde{H} = FG(X_H)$. Therefore $H = \langle X_H \rangle$.

To complete the proof it remains to show that the normal closure N_H of R_H in $FG(X_H) = \widetilde{H}$ is equal to $\widetilde{H} \cap N$.

Let $v \in V(\Gamma)$ such that $Q_v \neq \emptyset$. Let $q \in Q_v$. Therefore $\phi(t_v q \overline{t_v})$ is freely equivalent to the path $t_v q \overline{t_v}$. Thus

$$lab(\phi(t_v q \overline{t_v})) =_F lab(t_v q \overline{t_v}) \equiv lab(t_v) lab(q) lab(t_v)^{-1} \in N.$$

On the other hand, the path $t_v q \overline{t_v}$ is closed at v_0, hence $lab(t_v q \overline{t_v}) \in \widetilde{H}$. Thus $lab(\phi(t_v q \overline{t_v})) \in \widetilde{H} \cap N$. Therefore $R_H \subseteq \widetilde{H} \cap N$.

For all $y \in \widetilde{H}$, there exist a closed path $s \in \Gamma$ starting at v_0 with $lab(s) =_F \widetilde{H}$. By the definition of R_H, for all $r \in R_H$ there exist a path $t_v q \overline{t_v} \subseteq \Gamma$ closed at v_0 such that $lab(t_v q \overline{t_v}) =_F r$. Hence the path $s(t_v q \overline{t_v})\overline{s}$ is closed at v_0 in Γ. Thus $lab(s(t_v q \overline{t_v})\overline{s}) =_F yry^{-1} \in \widetilde{H}$. Moreover, $lab(s(t_v q \overline{t_v})\overline{s}) \equiv lab(st_v) lab(q) (lab(st_v))^{-1} \in N$. Therefore $N_H \subseteq \widetilde{H} \cap N$.

Assume now that $w \in \widetilde{H} \cap N$. Since $w \in \widetilde{H}$, there exists a freely reduced path p in Γ closed at v_0 with $lab(p) =_F w$ ([25, 37]). Let $p = p_1 \cdots p_k$ be its decomposition into maximal monochromatic paths p_i with $lab(p_i) \equiv w_i \in G_{l_i}$ ($1 \leq i \leq k$ and $l_i \in \{1, 2\}$).

Since $w \in N$, $w =_G 1$. Therefore, by the Normal Form Theorem for free products with amalgamation (IV.2.6 in [35]), there exists $1 \leq i \leq k$ such that $w_i \in A \cap G_{l_i}$. The proof is by induction on the number k of the maximal monochromatic subpaths of the path p. Without loss of generality, simplifying the notation, we let $l_i = 1$.

Assume first that $w_i =_{G_1} 1$. Since Γ is G-based, the subpath p_i is closed at $\iota(p_i) = \tau(p_i)$.

Let $v_j = \tau(p_j)$ and let $t_j = t_{v_j} \subseteq T$ ($1 \leq j \leq k$). Thus $v_{i-1} = v_i$ and $t_{i-1} = t_i$. Let $t = p_1 \cdots p_{i-1}$. See Figure 3 (a).

Hence the path p can be obtained by free reductions from the following path

$$\left((t\overline{t}_{i-1})(t_{i-1} p_i \overline{t_{i-1}})(t_{i-1}\overline{t}) \right) (t p_{i+1} \cdots p_k).$$

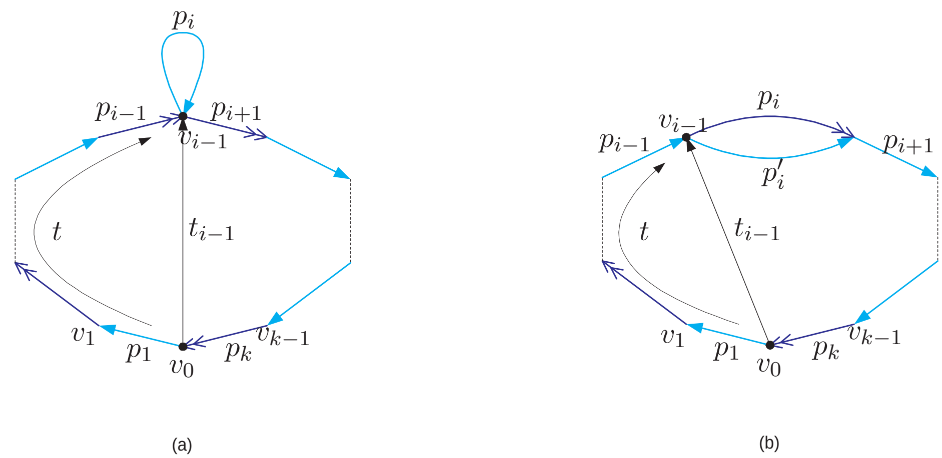

FIGURE 3.

Thus $lab(t\overline{t}_{i-1}) \in \widetilde{H}$, and the number of the maximal monochromatic subpaths of the path

$$tp_{i+1}\cdots p_k = p_1\cdots p_{i-2}(p_{i-1}p_{i+1})p_{i+2}\cdots p_k$$

is $k-2$. Therefore, by the inductive assumption, $lab(tp_{i+1}\cdots p_k) \in N_H$. To get the desired conclusion it remains to show that $lab(t_{i-1}p_i\overline{t_{i-1}}) \in N_H$.

Since $w_i =_{G_1} 1$, we have $lab(p_i) \in N_1$, where N_1 is the normal closure of R_1 in $F_1 = FG(X_1)$. Therefore $lab(p_i) =_{F_1} (z_1s_1z_1^{-1})\cdots(z_ms_mz_m^{-1})$, where $z_j \in F_1$ and $s_j \in R_1$ $(1 \le j \le m)$.

Let C_i be the X_1-monochromatic component of Γ such that $p_i \subseteq C_i$. Since Γ is a precover of G, C is $X_1^{\pm}$-saturated. Hence p_i is a free reduction of the path $p_i' \subseteq C$ such that $lab(p_i') \equiv (z_1s_1z_1^{-1})\cdots(z_ms_mz_m^{-1})$. Since Γ is G-based, the subpaths of p_i' labelled by s_j $(1 \le j \le m)$ are closed. Therefore p_i' has the following decomposition

$$p_i' = (c_1q_1\overline{c_1})\cdots(c_mq_m\overline{c_m}),$$

where $lab(c_j) \equiv z_j$ and $lab(q_j) \equiv s_j$ $(1 \le j \le m)$. Thus the path $t_{i-1}p_i\overline{t_{i-1}}$ can be obtained by free reductions from the path

$$\left(t_{i-1}(c_1q_1\overline{c_1})\overline{t_{i-1}}\right)\cdots\left(t_{i-1}(c_mq_m\overline{c_m})\overline{t_{i-1}}\right).$$

For each $1 \le j \le m$, the path $t_{i-1}(c_jq_j\overline{c_j})\overline{t_{i-1}}$ is a free reduction of the path

$$(t_{i-1}c_1\overline{t_{\tau(c_j)}})(t_{\tau(c_j)}q_j\overline{t_{\tau(c_j)}})(\overline{t_{i-1}c_j\overline{t_{\tau(c_j)}}}).$$

Since $lab(t_{i-1}c_j\overline{t_{\tau(c_j)}}) \in \widetilde{H}$ and $lab(\phi(t_{\tau(c_j)}q_j\overline{t_{\tau(c_j)}})) \in R_H$ we conclude that $lab(t_{i-1}(c_jq_j\overline{c_j})\overline{t_{i-1}}) \in N_H$ $(1 \le j \le m)$. Therefore $lab(t_{i-1}p_i\overline{t_{i-1}}) \in N_H$. We are done.

Assume now that $1 \neq_G w_i \in A \cap G_1$.

Since for all $1 \leq j \leq k$ the vertices $v_j = \tau(p_j)$ are bichromatic, and because the graph Γ is compatible, there exists a X_2-monochromatic path p'_i in Γ such that $\iota(p'_i) = \iota(p_i)$, $\tau(p'_i) = \tau(p_i)$ and $lab(p'_i) =_G lab(p_i)$. See Figure 3 (b). Hence the path p can be obtained by free reductions from the following path

$$\Big((t\overline{t}_{i-1})(t_{i-1}p_i\overline{p'_i}\overline{t_{i-1}})(t_{i-1}\overline{t}) \Big) \left(tp'_ip_{i+1}\cdots p_k\right).$$

Thus $lab(t\overline{t}_{i-1}) \in \widetilde{H}$, and the number of the maximal monochromatic subpaths of the path

$$tp'_ip_{i+1}\cdots p_k = p_1\cdots p_{i-2}(p_{i-1}p'_ip_{i+1})p_{i+2}\cdots p_k$$

is $k-2$. Therefore, by the inductive assumption, $lab(tp'_ip_{i+1}\cdots p_k) \in N_H$. To get the desired conclusion it remains to show that $lab(t_{i-1}(p_i\overline{p'_i})\overline{t_{i-1}}) \in N_H$.

Let $lab(p_i) =_{G_1} a_1\cdots a_m$, where a_j are generators of $A \cap G_1$. Let b_j be corresponding generators of $A \cap G_2$ such that $a_j =_G b_j$ and $a_j{b_j}^{-1} \in R$ $(1 \leq j \leq m)$. Note that

$$(a_1\cdots a_m)(b_1\cdots b_m)^{-1} =_F$$

$$=_F \left(a_1{b_1}^{-1}\right)\left(b_1\left(a_2{b_2}^{-1}\right)b_1^{-1}\right)\cdots\left(b_1\cdots b_{m-1}\left(a_mb_m^{-1}\right)b_{m-1}^{-1}\cdots b_1^{-1}\right).$$

Since monochromatic components of Γ are $X_i^{\pm}$-saturated ($i \in \{1,2\}$), and because $\iota(p_i) \in VB(\Gamma)$, there exist paths γ_1 and δ_1 such that $\iota(\gamma_1) = \iota(p_i) = \iota(\delta_1)$ and $lab(\gamma_1) \equiv a_1$, $lab(\delta_1) \equiv b_1$. Since Γ is compatible, $\tau(\gamma_1) = \tau(\delta_1) \in VB(\Gamma)$. Thus there exist paths γ_2 and δ_2 such that $\iota(\gamma_2) = \tau(\gamma_1) = \iota(\delta_2)$ and $lab(\gamma_2) \equiv a_2$, $lab(\delta_2) \equiv b_2$. Since Γ is compatible, $\tau(\gamma_2) = \tau(\delta_2) \in VB(\Gamma)$.

Continuing in this manner one can construct such paths γ_j, δ_j for all $1 \leq j \leq m$. Thus p_i and p'_i are free reductions of the paths $\gamma_1\cdots\gamma_m$ and $\delta_1\cdots\delta_m$, respectively. Hence the path $p_i\overline{p'_i}$ can be obtained by free reductions from the path

$$\left(\gamma_1\overline{\delta_1}\right)\left(\delta_1\left(\gamma_2\overline{\delta_2}\right)\overline{\delta_1}\right)\cdots\left(\delta_1\cdots\delta_{m-1}\left(\gamma_m\overline{\delta_m}\right)\overline{\delta_{m-1}}\cdots\overline{\delta_1}\right).$$

Therefore the path $t_{i-1}(p_i\overline{p'_i})\overline{t}_{i-1}$ is a free reduction of

$$\left(t_{i-1}(\gamma_1\overline{\delta_1})\overline{t}_{i-1}\right)\cdots\left(t_{i-1}(\delta_1\cdots\delta_{m-1}\left(\gamma_m\overline{\delta_m}\right)\overline{\delta_{m-1}}\cdots\overline{\delta_1})\overline{t}_{i-1}\right).$$

For each $1 \leq j \leq m-1$, the path $t_{i-1}(\delta_1\cdots\delta_{j-1}\left(\gamma_j\overline{\delta_j}\right)\overline{\delta_{j-1}}\cdots\overline{\delta_1})\overline{t}_{i-1}$ is a free reduction of the path

$$\left(t_{i-1}\delta_1\cdots\delta_{j-1}\overline{t_{\iota(\gamma_j)}}\right)\left(t_{\iota(\gamma_j)}(\gamma_j\overline{\delta_j})\overline{t_{\iota(\gamma_j)}}\right)\left(t_{\iota(\gamma_j)}\overline{\delta_{j-1}}\cdots\overline{\delta_1}\overline{t_{i-1}}\right).$$

Since $lab\left(\phi\left(t_{\iota(\gamma_j)}(\gamma_j\overline{\delta_j})\overline{t_{\iota(\gamma_j)}}\right)\right) \in R_H$ and $lab(t_{i-1}\delta_1 \cdots \delta_{j-1}\overline{t_{\iota(\gamma_j)}}) \in \widetilde{H}$, we conclude that for each $1 \leq j \leq m-1$

$$lab(t_{i-1}(\delta_1 \cdots \delta_{j-1}\left(\gamma_j\overline{\delta_j}\right)\overline{\delta_{j-1}} \cdots \overline{\delta_1})\overline{t}_{i-1}) \in N_H.$$

Therefore $lab(t_{i-1}(p_i\overline{p'_i})\overline{t_{i-1}}) \in N_H$. We are done.

$\diamond$

Corollary 5.3. *Let (Γ, v_0) be a finite precover of G. Then there exists an algorithm which computes a subgroup of G determined by (Γ, v_0), that is computes a finite group presentation of $H = Lab(\Gamma, v_0)$.*

Proof. We compute the sets X_H and R_H according to their definitions. These sets are finite, because the graph Γ is finite. By Theorem 5.2, $H = gp\langle X_H \mid R_H\rangle$.

$\diamond$

Corollary 5.4. *Let $h_1, \ldots h_n \in G$. Then there exists an algorithm which computes a finite group presentation of the subgroup $H = \langle h_1, \ldots, h_n\rangle$ in G (not necessary with respect to $\{h_1, \cdots, h_n\}$) .*

Proof. We first construct the graph $(\Gamma(H), v_0)$, using the generalized Stallings' folding algorithm. By Theorem 4.1 (2), this graph is a finite precover of G. Now we proceed according to Corollary 5.3.

$\diamond$

Corollary 5.5. *Amalgams of finite groups are coherent.*

Remark 5.6. As is well known, the Reidemeister-Schreier method yields a presentation of a subgroup H which is usually not in a useful form. Namely, some of the generators are redundant and can be eliminated, while some of the relators can be simplified. In order to improve (to simplify) this presentation, one can apply the Tietze transformation. An efficient version of such a simplification procedure was developed in [21, 22]. $\diamond$

Example 5.7. Let $G = gp\langle x, y | x^4, y^6, x^2(y^3)^{-1}\rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Recall that G is isomorphic to $SL(2, \mathbb{Z})$ under the homomorphism

$$x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; y \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let $H = \langle xyx^{-1}, yxy^{-1}\rangle$ be a subgroup of G. The subgroup graph $\Gamma(H)$ constructed by the generalized Stallings' folding algorithm is presented on Figure 4.

We apply to $\Gamma(H)$ the algorithm described along with the proof of Corollary 5.4.

We first compute X_H according to (1):

$$h_1 = xyx^{-1},\ h_2 = x^2,\ h_3 = yxy^{-1},\ h_4 = y^3.$$

The computation of R_H according to (3) consists of the following steps.

$$\phi'(x^4) = (h_2)^2,\ \phi'(y^6) = (h_4)^2,\ \phi'(x^2(y^3)^{-1}) = h_2(h_4)^{-1}.$$

$$\phi'(x(x^4)x^{-1}) = (h_2)^2,\ \phi'(x(y^6)x^{-1}) = (h_1)^6,\ \phi'(x(x^2(y^3)^{-1})x^{-1}) = h_2(h_1)^{-3}.$$

$$\phi'(y(x^4)y^{-1}) = (h_3)^4,\ \phi'(y(y^6)y^{-1}) = (h_4)^2,\ \phi'(y(x^2(y^3)^{-1})y^{-1}) = h_3^2(h_4)^{-1}.$$

Therefore $H = gp\langle h_1, h_3 \mid h_1^6, h_3^4, h_1^3 = h_3^2\rangle$.

$\diamond$

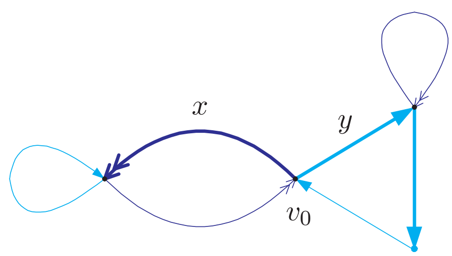

FIGURE 4. The bold edges of the graph $\Gamma(H)$ correspond to a spanning tree T

Complexity. Let m be the sum of the lengths of the words $h_1, \dots h_n$. By Theorem 4.1 (4), the generalized Stallings' algorithm computes $(\Gamma(H), v_0)$ in time $O(m^2)$.

The construction of X_H, which is a free basis of $\widetilde{H} = \varphi_1\,(lab(Loop(\Gamma(H), v_0))$, takes $O(|E(\Gamma(H))|^2)$, by [4]. Since, by Theorem 4.1 (4), $|E(\Gamma(H))|$ is proportional to m, the computation of X_H takes $O(m^2)$.

To construct the set Q_H we try to read each one of the defining relators of G at each one of the vertices of the graph $\Gamma(H)$. It takes at most

$$|R| \cdot |V(\Gamma(H)| \cdot \left(\sum_{v \in V(\Gamma(H))} deg(v) \right).$$

Since $\sum_{v \in V(\Gamma(H))} deg(v) = 2|E(\Gamma(H)|$ and because, by our assumption, the presentation of G is given and it is not a part of the input, the computation of the set Q_H takes $O(|V(\Gamma(H)| \cdot |E(\Gamma(H)|)$. Since, by Theorem 4.1 (4), $|V(\Gamma(H)| = O(m)$, it takes $O(m^2)$.

The rewriting process which yield the set of relators R_H takes at most $|V(\Gamma(H)| \cdot \left(\sum_{r \in R} |r|\right)$ which is $O(|V(\Gamma(H)|)$.

Thus the complexity of the restricted Reidemeister-Schreier process given by Corollary 5.4 is $O(m^2)$.

6. The Freeness Problem

A freeness of subgroups is one of the fundamental questions of combinatorial and geometric group theory. The classical results in this issue include the Nielsen-Schreier subgroup theorem for free groups, the corollary of Kurosh subgroup theorem and the Freiheitssatz of Magnus.

Namely, subgroups of free groups are free (I.3.8, [35]). A subgroup of a free product which has a trivial intersection with all conjugates of the factors is free ([35], p.120). A subgroup H of an one-relator group $G = gp\langle X \mid r = 1 \rangle$, where r is *cyclically freely reduced*, is free if H is generated by a subset of X which omits a generator occurring in r (II.5.1, [35]).

Results concerning amalgamated free products follow from the Neumann's subgroup theorem.

Theorem 6.1 (H.Neumann, IV.6.6 [35])**.** *Let $G = G_1 *_A G_2$ be a non-trivial free product with amalgamation. Let H be a finitely generated subgroup of G such that all conjugates of H intersect A trivially.*

Then $H = F * (*_j\ g_j H_j g_j^{-1})$, where F is a free group and each H_j is the intersection of a subgroup of H with a conjugate of a factor of G.

Corollary 6.2 (IV.6.7 [35])**.** *Let $G = G_1 *_A G_2$ be a non-trivial free product with amalgamation. If H is a finitely generated subgroup of G which has trivial intersection with all conjugates of the factors, G_1 and G_2, of G, then H is free.*

It turns out (Lemma 6.3) that the triviality of the intersections between H and conjugates of the factors, G_1 and G_2, of G can be detected from the subgroup graph $\Gamma(H)$ constructed by the generalized Stallings' folding algorithm, when $G = G_1 *_A G_2$ is an amalgam of finite groups. Therefore, by Corollary 6.2, the freeness of H is decidable via its subgroup graph.

We consider the *freeness problem* to be one which asks to verify if a subgroup of a given group G is free. Clearly, the freeness problem is solvable in amalgams of finite groups.

Below we introduce a polynomial time algorithm (Corollary 6.7) that employs subgroup graphs constructed by the generalized Stallings' algorithm to solve the freeness problem. A complexity analysis of the algorithm is given at the end of the section.

Lemma 6.3. *Let H be a finitely generated subgroup of an amalgam of finite groups $G = G_1 *_A G_2$.*

Then H has a trivial intersection with all conjugates of the factors of G if and only if each X_i-monochromatic component C of $\Gamma(H)$ is isomorphic to $Cayley(G_i)$, for all $i \in \{1, 2\}$. Equivalently, by Lemma 4.6, if and only if $Lab(C, v) = \{1\}$ for each X_i-monochromatic component C of $\Gamma(H)$ ($v \in V(C)$).

Proof. Assume first that there exists a X_i-monochromatic component C of $\Gamma(H)$ ($i \in \{1, 2\}$) which is not isomorphic to $Cayley(G_i)$. Thus, by Lemma 4.6, (C, ϑ) is not isomorphic to $Cayley(G_i, S, S \cdot 1)$, where is $\vartheta \in V(C)$ and $\{1\} \neq S \leq G_i$.

Let $1 \neq_G w \in S$. Then there exists a path q in C closed at ϑ such that $lab(q) \equiv w$. Let p be an approach path in $\Gamma(H)$ from $\iota(p) = v_0$ to $\tau(p) = \vartheta$. Let $u \equiv lab(p)$.

The path $pq\overline{p}$ is closed at v_0 in $\Gamma(H)$. Hence $lab(pq\overline{p}) \in H$. Therefore

$$lab(pq\overline{p}) =_G uwu^{-1} \in H \cap uLab(C, \vartheta)u^{-1} = H \cap uSu^{-1}.$$

Since $w \neq_G 1$, we have $uwu^{-1} \neq_G 1$ and hence $H \cap uSu^{-1} \neq \{1\}$.

Assume now that there exists $\{1\} \neq S \leq G_i$ ($i \in \{1, 2\}$) such that $H \cap uSu^{-1} \neq \{1\}$, where $u \in G$. Let $1 \neq_G h \in H \cap uSu^{-1}$. Thus $h =_G ugu^{-1}$, where $1 \neq_G g \in S$. Without loss of generality we can assume that the words u and g are normal.

If the word ugu^{-1} is in normal form, then there exist a path p in $\Gamma(H)$ closed at v_0 such that $lab(p) \equiv ugu^{-1}$. Thus there is a decomposition $p = p_1 p_2 \overline{p_1}$ (because $\Gamma(H)$ is G-based, so it is a well-labelled graph), where $lab(p_1) \equiv u$ and $lab(p_2) \equiv g$. Let C be a X_i-monochromatic component of $\Gamma(H)$ such that $p_2 \subseteq C$ and let $v = \tau(p_1)$. Hence $g \equiv lab(p_2) \in Lab(C, v) \leq G_i$. Thus $Lab(C, v) \neq \{1\}$. Equivalently, by Lemma 4.6, C is not isomorphic to $Cayley(G_i)$.

Assume now that the word ugu^{-1} is not in normal form. Let $(u_1, \ldots, u_k)$ be a normal decomposition of u. Since $g \in G_i$, its normal decomposition is (g). Hence the normal decomposition of ugu^{-1} has the form

$$(u_1, \ldots, u_{j-1}, w, u_{j-1}^{-1}, \ldots, u_1^{-1}),$$

where $w =_G u_j \ldots u_k g u_k^{-1} \ldots u_j^{-1} \in G_l \setminus A$ and $u_{j-1} \in G_m \setminus A$ ($1 \leq l \neq m \leq 2$).

Let $u' \equiv u_1 \ldots u_{j-1}$. Then $h =_G u'w(u')^{-1}$, while the word $u'w(u')^{-1}$ is in normal form and $w \in G_l$, $l \in \{1, 2\}$. Hence, by arguments similar to those used in the previous case, we are done.

$\diamond$

Theorem 6.4. *Let H be a finitely generated subgroup of an amalgam of finite groups $G = G_1 *_A G_2$.*

Then H is free if and only if each X_i-monochromatic component of $\Gamma(H)$ is isomorphic to $Cayley(G_i)$, for all $i \in \{1, 2\}$.

Proof. The statement follows immediately from Corollary 6.2 and Lemma 6.3. ◇

Combining Lemma 6.3 with the Torsion Theorem for amalgamated free products we get Corollary 6.6.

Theorem 6.5 (Torsion Theorem, IV.2.7, [35])**.** *Every element of finite order in $G = G_1 *_A G_2$ is a conjugate of an element of finite order in G_1 or G_2.*

Corollary 6.6. *Let H be a finitely generated subgroup of an amalgam of finite groups $G = G_1 *_A G_2$.*

Then H is torsion free if and only if each X_i-monochromatic component of $\Gamma(H)$ is isomorphic to $Cayley(G_i)$, for all $i \in \{1, 2\}$.

Corollary 6.7. *Let $h_1, \ldots, h_k \in G$. Then there exists an algorithm which decides whether or not the subgroup $H = \langle h_1, \ldots, h_k \rangle$ is a free subgroup of G.*

Proof. We first construct the graph $\Gamma(H)$, using the generalized Stallings' folding algorithm.

Now, for each X_i-monochromatic component C of $\Gamma(H)$ we verify if C is isomorphic to $Cayley(G_i)$ $(i \in \{1, 2\})$. It can be easily done by checking the number of vertices of C: $|V(C)| = |G_i|$ if and only if C is isomorphic to $Cayley(G_i)$.

By Theorem 6.4, H is free if and only if each monochromatic component of $\Gamma(H)$ is isomorphic to the Cayley graph of an appropriate factor of G.

◇

Remark 6.8. If H is free then its free basis can be computed using the restricted Reidemeister-Schreier procedure (Corollary 5.4) followed by a simplification process based on Tietze transformation. For an effective version of a simplification procedure when redundant generators are eliminated consequently using a substring search technique see [21, 22]. ◇

Example 6.9. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Let H_1 and H_2 be finitely generated subgroups of G such that

$$H_1 = \langle xy \rangle \text{ and } H_2 = \langle xy^2, yxyx \rangle.$$

The graphs $\Gamma(H_1)$ and $\Gamma(H_2)$ on Figure 5 are the subgroup graphs of H_1 and H_2, respectively, constructed by the generalized Stallings' folding

algorithm. See Example A.2 from Appendix for the detailed construction of these graphs.

Applying the above algorithm to the graphs $\Gamma(H_1)$ and $\Gamma(H_2)$, we conclude that H_2 is not free, while $H_1 = FG(\{xy\})$. $\diamond$

FIGURE 5.

Complexity. Let m be the sum of the lengths of the words $h_1, \ldots h_k$. By Theorem 4.1 (4), the complexity of the construction of $\Gamma(H)$ is $O(m^2)$.

The detecting of monochromatic components in this graph takes $O(|E(\Gamma(H))|)$. Since, by our assumption, all the essential information about A, G_1 and G_2 is given and it is not a part of the input, verifications concerning a particular monochromatic component of $\Gamma(H)$ take $O(1)$. Therefore to do such verifications for all monochromatic component of $\Gamma(H)$ takes $O(|E(\Gamma(H))|)$. Since, by Theorem 4.1 (4), $|E(\Gamma(H))|$ is proportional to m, the complexity of the "freeness" detecting presented along with the proof of Corollary 6.7 is $O(m^2)$, that is it is quadratic in the size of the input.

If the subgroup H is given by the graph $\Gamma(H)$, then to verify that H is a free subgroup of G takes $O(|E(\Gamma(H))|)$. That is the "freeness" algorithm is even linear in the size of the input.

7. The Finite Index Problem

One of the first natural computational questions regarding subgroups is to compute the index of the subgroup in the given group, the *finite index problem*.

As is well known ([4, 25]), this problem is easily solvable via subgroup graphs in the case of free groups. Recall that a subgroup H of a free group $FG(X)$ has finite index if and only if its subgroup graph Γ_H constructed by the Stallings' folding algorithm is *full*, i.e. *complete*, i.e. $X^{\pm}$-*saturated*. That is for each vertex $v \in V(\Gamma_H)$ and for each $x \in X^{\pm}$ there exists an edge which starts at v and which is labelled by x.

In [50] similar results were obtained for finitely generated subgroups of certain Coxeter groups and surface groups of an extra-large type.

In general, the index $[G : H]$ equals to the *sheet number* of the covering space, corresponding to the subgroup H, of the standard 2-complex representing the group G ([54]). Thus if G is finitely presented, the index of H in G is finite if and only if the 1-skeleton of the corresponding covering space is finite. That is if and only if the relative Cayley graph, $Cayley(G, H)$, is finite.

By Theorem 4.1 (2) and Corollary 4.7, a subgroup graph $(\Gamma(H), v_0)$ is a subgraph of $(Cayley(G, H), H \cdot 1)$. It turns out that there exists a strong connection between the index of H in G and how "saturated" the graph $\Gamma(H)$ is. We describe this connection in Theorem 7.1 and use it to solve the *finite index problem* in amalgams of finite groups (Corollary 7.2).

The complexity analysis of the presented algorithm is given at the end of the section.

Theorem 7.1. *Let H be a finitely generated subgroup of an amalgam of finite groups $G = G_1 * G_2$.*

Then $[G : H] < \infty$ if and only if $\Gamma(H)$ is $X^{\pm}$-saturated.

Proof. The "if" direction is clear. Indeed, if $\Gamma(H)$ is $X^{\pm}$-saturated then, by Lemma 4.6, $\Gamma(H)$ is isomorphic to $Cayley(G, H, H \cdot 1)$. Since, by Theorem 4.1, the graph $\Gamma(H)$ is finite, $Cayley(G, H, H \cdot 1)$ is a finite graph. Hence $[G : H] = |V(Cayley(G, H))| < \infty$.

To prove the opposite direction we assume that $\Gamma(H)$ is not $X^{\pm}$-saturated. Note that since $\Gamma(H)$ is a precover of $Cayley(G, H)$, any of its monochromatic component are either $X_1^{\pm}$-saturated or $X_2^{\pm}$-saturated. Thus every bichromatic vertex of Γ is $X^{\pm}$-saturated and each monochromatic vertex is either $X_1^{\pm}$-saturated or $X_2^{\pm}$-saturated.

Let v be a X_1-monochromatic vertex of Γ. Then by Lemma 4.10, there is a path p in normal form such that $\iota(p) = v_0$, $\tau(p) = v$, and $w \equiv lab(p)$ is a word in normal form. Let $(w_1, \ldots, w_n)$ be a normal decomposition of w. Then there is $x \in X_2 \setminus A$ [4], such that $(w_1, \ldots, w_n, x)$ represents a word $w' \in G$ in normal form. Now if $w_1 \in G_1$ (more precisely, $w_1 \in G_1 \setminus A$, since w is a normal word) or if $w_1 \in G_2$ but $xw_1 \in G_2 \setminus A$ then $(w')^n$ is in normal form for all $n \geq 1$.

Otherwise $xw_1 \in G_2 \cap A$. Then there exists $y \in X_1 \setminus A$, such that $(w_1, \ldots, w_n, x, y)$ represents a word $w'' \in G$ in normal form and $(w'')^n$ is in normal form for all $n \geq 1$. But neither w' nor w'', and hence neither $(w')^n$

[4] We assume that A is a proper subgroup of G_1 and of G_2, otherwise the amalgam $G_1 *_A G_1$ is a finite group and all computations are trivial in our context

nor $(w'')^n$ (for all $n \geq 1$) label a path closed at v_0 in $\Gamma(H)$. Thus $(w')^n \notin H$ and $(w'')^n \notin H$, for all $n \geq 1$.

The existence of such elements shows that H has infinite index in G. Indeed, for all $n_1 \neq n_2$ and $g \in \{w', w''\}$ we have $H(g)^{n_1} \neq H(g)^{n_2}$, because otherwise $(g)^{n_1 - n_2} \in H$. Thus, without loss of generality we can assume that $n_1 > n_2$, then $n_1 - n_2 \geq 1$ and we get a contradiction.

$\diamond$

Corollary 7.2. *Let $h_1, \ldots h_n \in G$. Then there exists an algorithm which computes the index of the subgroup $H = \langle h_1, \ldots, h_n \rangle$ in G.*

Proof. We first construct the graph $\Gamma(H)$, using the generalized Stallings' folding algorithm.

Then we verify if this graph is $(X_1 \cup X_2)^{\pm}$-saturated. If no, the subgroup H has infinite index in G, by Theorem 7.1. Otherwise, the index of H in G is finite and $[G : H] = |V(\Gamma(H))|$.

$\diamond$

Complexity. Let m be the sum of the lengths of the words $h_1, \ldots h_n$. By Theorem 4.1 (4), the generalized Stallings' algorithm computes $(\Gamma(H), v_0)$ in time $O(m^2)$. By the proof of Corollary 7.2, the detecting of the index takes time proportional to $|E(\Gamma(H)|$. (Indeed, for each vertex of $\Gamma(H)$ we have to check if it is bichromatic, which takes $\sum_{v \in V(\Gamma(H))} deg(v) = 2|E(\Gamma(H)|$.) Since, by Theorem 4.1 (4), $|E(\Gamma(H)| = O(m)$, the complexity of the algorithm given along with the proof of Corollary 7.2 is $O(m^2)$.

If the subgroup H is given by $(\Gamma(H), v_0)$ and not by a finite set of subgroup generators, then the above algorithm is even linear in the size of the graph.

Example 7.3. Let H_1 and H_2 be the subgroups considered in Example 6.9.

Analyzing the "saturation" of the graphs $\Gamma(H_1)$ and $\Gamma(H_2)$ illustrated on Figure 5, we see that $[G : H_1] = \infty$, while $[G : H_2] = 2$. $\diamond$

8. The Separability Problem

A group G is *subgroup separable*, or *LERF*, if given a finitely generated subgroup H of G and $g \notin H$ there exists a finite index subgroup $K \leq G$ with $H \leq K$ and $g \notin K$. We call K a *separating subgroup*. If one places a topology on G (called the *profinite topology* [20]), by taking the collection of finite index subgroups as a neighborhood basis of 1, then G is LERF if and only if all its finitely generated subgroups are closed.

LERF was introduced by M.Hall [19], who proved that free groups are LERF. This property is preserved by free products [8, 49], but it is not preserved by direct products: $F_2 \times F_2$ is not LERF [1]. Free products of LERF groups with finite amalgamation are LERF [1]. In general, the property is

not preserved under free products with infinite cyclic amalgamation [32, 47]. However amalgams of free groups over a cyclic subgroup are LERF [7], and, by [14], free products of a free group and a LERF group amalgamated over a cyclic subgroup maximal in the free factor are LERF as well.

Subgroup separability of some classes of hyperbolic groups was widely exploited in papers of Gitik [14, 15, 16]. Long and Reid [33, 34] studied this property in 3-manifold topology and in hyperbolic Coxeter groups. Results on subgroup separability for right-angle Coxeter groups and for Coxeter groups of extra-large type can be found in [12] and in [50], respectively. These papers include detailed algorithms which construct separating subgroups using graph-theoretic methods.

M.Hall property is closely connected with subgroup separability. A groups G is *M.Hall* if and only if each of its finitely generated subgroups is a free factor in a subgroup of finite index in G. M.Hall property of virtually free groups was deeply studied in works of Bogopolskii [5, 6], where a criterion to determine whether a virtually free group is M.Hall was given.

An algorithmic aspect of the LERF property can be formulated as the *separability problem*. It asks to find an algorithm which constructs a separating subgroup K for a given finitely generated subgroup H and $g \notin H$.

Let us emphasize that the knowledge that G has a solvable decision problem does not provide yet an effective procedure to solve this problem. Thus, on the one hand, since amalgams of finite groups are LERF, by the result of Allenby and Gregorac [1], the separability problem in this class of groups might be solvable. On the other hand, we are interested to find an efficient solution. Below we adopt some ideas of Gitik introduced in [14] to develop such an algorithm (given along with the proof of the Main Theorem (Theorem 8.1)). Our main result in this issue is summarized in the following theorem.

Theorem 8.1 (The Main Theorem)**.** *Let $G = G_1 *_A G_2$ be an amalgam of finite groups.*

The <u>separability problem</u> for G is solvable if one of the following holds

1. *A is cyclic,*
2. *A is malnormal in at least one of the factors G_1 or G_2,*
3. *$A \leq Z(G_i)$, for some $i \in \{1, 2\}$.*

   * *In particular, the separability problem is solvable if at least one of the factors (G_1 or G_2) is Abelian.*

Recall that given a finitely generated subgroup H of an amalgam of finite groups $G = G_1 *_A G_2$ the generalized Stallings' algorithm constructs the canonical subgroup graph $\Gamma(H)$ which is a (reduced) precover of G (Theorem 4.1 (2)). Thus in order to prove our Main Theorem we first show

that each finite precover (Γ, v_0) of G, when G satisfies one of the conditions $(1) - (3)$, can be embedded in a finite X_i-saturated precover (Γ', v_0) of G ($i \in \{1, 2\}$). Then we prove that such a precover can be embedded in a finite cover (Γ'', v_0) of G. Finally, we take $K = Lab(\Gamma'', v_0)$ to be the separating subgroup. This completes the proof of the Main Theorem.

Example 8.8 demonstrates the computation of the separating subgroup K for a given subgroup $H \leq G$.

The *amalgam* of labelled graphs Γ_1 and Γ_2 along Γ_0 denoted by $\Gamma_1 *_{\Gamma_0} \Gamma_2$, is the pushout of the following diagram in the category of labelled graphs:

$$\begin{array}{ccc} \Gamma_0 & \rightarrow & \Gamma_1 \\ \downarrow & \searrow & \downarrow \\ \Gamma_2 & \rightarrow & \Gamma_1 *_{\Gamma_0} \Gamma_2, \end{array}$$

where $i_1 : \Gamma_0 \rightarrow \Gamma_1$ and $i_2 : \Gamma_0 \rightarrow \Gamma_2$ are injective maps and none of the graphs need be connected. The amalgam depends on the maps i_1 and i_2, but we omit reference to them, whenever it does not cause confusion. It can be easily seen that amalgamation consists of taking the disjoint union of graphs and performing the identification prescribed by i_1 and i_2 and subsequent *foldings* (an identification of the terminal vertices of a pair of edges with the same origin and the same label) until a labelled graph is obtained [14, 53].

Lemma 8.2. *Let Γ be a finite precover of an amalgamated free product of finite groups $G = G_1 *_A G_2$. Then Γ can be embedded in a $X_1^{\pm}$-saturated precover of G with finitely many vertices.*

Proof. Any vertex of a graph well-labelled with $X_1^{\pm} \cup X_2^{\pm}$ has one of the following types:

- It is bichromatic.
- It is X_1-monochromatic.
- It is X_2-monochromatic.

Since Γ is a precover of G, the above types take the form (respectively):

- It is $X_1^{\pm} \cup X_2^{\pm}$-saturated.
- It is X_1-monochromatic and $X_1^{\pm}$-saturated.
- It is X_2-monochromatic and $X_2^{\pm}$-saturated.

The proof is by induction on the number of vertices of the third type. If no such vertices exist, then Γ is already $X_1^{\pm}$-saturated. Assume that Γ has m X_2-monochromatic vertices, and let v be one of them.

Let C be a X_2-monochromatic component, such that $v \in VM_2(C)$. Let $S = A_v$ be the *stabilizer of* v by the action of A on the vertices of C, that

is $A_v = \{x \in A \mid v \cdot x = v\} \leq A$, and let $A(v) = \{v \cdot x \mid x \in A\} \subseteq V(C)$ be the A-*orbit* of v.

Consider $Cayley(G_1, S, S \cdot 1)$. Thus $A_{S \cdot 1} = S = A_v$ and the A-orbit $A(S \cdot 1) = \{(S \cdot 1) \cdot x \mid x \in A\} = \{Sx \mid x \in A\} \subseteq V(Cayley(G_1, S))$ is isomorphic to $A(v)$. Hence, taking $\Gamma_v = \Gamma *_{\{v \cdot x = Sx \mid x \in A\}} Cayley(G_1, S)$, we get a finite compatible graph whose monochromatic components are covers of the factors G_1 or G_2. Therefore, by Corollary 4.12, Γ_v is a precover of Γ.

Since $A_{S \cdot 1} = S = A_v$, the only identifications in Γ_v are between vertices of $A(v)$ and $A(S \cdot 1)$. Since these are sets of monochromatic vertices of different colors, no foldings are possible in Γ_v. Hence the graphs Γ and $Cayley(G_1, S)$ embed in Γ_v. Thus the images in Γ_v of the vertices of $A(v)$ (equivalently, of $A(S \cdot 1)$) are bichromatic vertices, while the chromacity of the images of other vertices of Γ and $Cayley(G_1, S)$ remains unchanged. Hence

$$|VM_2(\Gamma_v)| = |VM_2(\Gamma)| - |A(v)| < m.$$

Therefore Γ_v is a finite precover of G with $|VM_2(\Gamma_v)| < m$ such that Γ embeds in Γ_v. This completes the inductive step.

$\diamond$

Remark 8.3. By the symmetric arguments if the conditions of Lemma 8.2 hold then Γ can be embedded in a $X_i^{\pm}$-saturated precover of G ($i \in \{1, 2\}$) with finitely many vertices. $\diamond$

The proof of Lemma 8.2 yields the following technical result, which we employ later to produce X_i-saturated precovers ($i \in \{1, 2\}$).

Corollary 8.4. *Let $G = G_1 *_A G_2$ be an amalgam of finite groups.*

Let Γ_i be a finite precover of G (not necessary connected) and let $v_i \in VM_i(\Gamma_i)$ ($i \in \{1, 2\}$).

If $A_{v_1} = A_{v_2}$ then $A(v_1) \simeq A(v_2)$, and $\Gamma = \Gamma_1 *_{\{v_1 \cdot a = v_2 \cdot a \mid a \in A\}} \Gamma_2$ is a finite precover of G such that the graphs Γ_1 and Γ_2 embed into the graph Γ.

Now we consider Γ to be a finite $X_\beta^{\pm}$-saturated precover of G ($\beta \in \{1, 2\}$), where $G = G_1 *_A G_2$ is an amalgamated free product of finite groups. In the consequent lemmas, it is showen that if one of the conditions from Theorem 8.1 is satisfied then Γ can be embedded into a finite cover of G.

Since the graph Γ is $X_\beta^{\pm}$, any vertex of Γ is either bichromatic or X_β-monochromatic. Moreover, the graph Γ is compatible, as a precover of G. Hence any A-orbit consists of the vertices of the same type. Therefore the set of X_β-monochromatic vertices of Γ can be viewed as a disjoint union of distinct A-orbits. This enables us to consider the following notation.

For each $v \in VM_\beta(\Gamma)$ we set n_v to be the number of vertices in the A-orbit of v, that is $n_v = |A(v)|$. Recall that

$$A_v \leq A, \ A(v) \simeq A/A_v, \ \text{thus} \ |A| = |A(v)||A_v|.$$

Let $n(\Gamma) = \{n_v | v \in VM_\beta(\Gamma)\}$. For each $n \in n(\Gamma)$, assume that Γ has m different A-orbits, each containing n X_β-monochromatic vertices. Let $\{v_i | 1 \leq i \leq m\}$ be the set of representatives of these orbits. Denote $S_i = A_{v_i}$. Then for all $1 \leq i \leq m$, $|S_i| = \frac{|A|}{n}$.

Assume that A has r distinct subgroups S_j $(1 \leq j \leq r)$ of order $\frac{|A|}{n}$ and assume that Γ has m_j representatives of distinct orbits $v_j \in VM_\beta(\Gamma)$ with $A_{v_j} = S_j$. Hence $\sum_{j=1}^r m_j = m$.

With the above notation, we formulate Lemmas 8.5, 8.6 and 8.7.

Lemma 8.5. *If A is a center subgroup of G_α (that is $A \leq Z(G_\alpha)$ [5]), then any finite $X_\beta^\pm$-saturated precover of G can be embedded in a cover of G with finitely many vertices ($1 \leq \beta \neq \alpha \leq 2$).*

Proof. The proof is by induction on $|n(\Gamma)|$.

Since $A \leq Z(G_\alpha)$, S_j is normal in G_α for all $1 \leq j \leq r$. Therefore for each vertex $u \in V(Cayley(G_\alpha, S_j))$, we have $A_u = S_j$. Indeed,

$$A_u = Lab(Cayley(G_\alpha, S_j), u) \cap A = g^{-1} S_j g \cap A = S_j \cap A = S_j,$$

where $g \in G_\alpha$, such that $(S_j \cdot 1) \cdot g = u$. Thus distinct A-orbits of vertices in $Cayley(G_\alpha, S_j)$ are isomorphic to each other and have length n. Their number is equal to

$$\frac{|V(Cayley(G_\alpha, S_j))|}{n} = \frac{|G_\alpha|}{|S_j|} : \frac{|A|}{|S_j|} = \frac{|G_\alpha|}{|A|} = [G_\alpha : A].$$

Let $t = [G_\alpha : A]$. Let Γ_1 be the disjoint union of t isomorphic copies of Γ and let Γ_2 be the disjoint union of m_j isomorphic copies of $Cayley(G_\alpha, S_j)$, for all $1 \leq j \leq r$. Then both Γ_1 and Γ_2 have tm_j distinct isomorphic A-orbits of length n.

Let $\{w_{ji} \mid 1 \leq i \leq tm_j, \ 1 \leq j \leq r\}$ and $\{u_{ji} \mid 1 \leq i \leq tm_j, \ 1 \leq j \leq r\}$ be the sets of representatives of these orbits in Γ_1 and in Γ_2, respectively. Thus $A_{w_{ji}} = S_j = A_{u_{ji}}$, for all $1 \leq i \leq tm_j$ and $1 \leq j \leq r$. Let Γ' be the amalgam of Γ_1 and Γ_2 over these sets of vertices,

$$\Gamma' = \Gamma_1 *_{\{w_{ji} \cdot a = u_{ji} \cdot a \ | \ a \in A\}} \Gamma_2.$$

By Corollary 8.4, Γ' is a finite precover of G such that the graphs Γ_1 and Γ_2 embed in it. Therefore the graph Γ embeds in Γ' as well. Moreover, by construction, the graph Γ' is $X_\beta^\pm$-saturated, and $n(\Gamma') = n(\Gamma) \setminus \{n\}$. Thus Γ' satisfies the inductive assumption. We are done.

[5] Recall that the *center of* G is the subgroup $Z(G) = \{g \in G \mid gx = xg, \ \forall x \in G\}$.

◇

Lemma 8.6. *If A is a malnormal subgroup of G_α then any finite $X_\beta^{\pm}$-saturated precover of G can be embedded in a cover of G with finitely many vertices ($1 \leq \beta \neq \alpha \leq 2$).*

Proof. The proof is by induction on $|n(\Gamma)|$.

Since A is malnormal in G_α, for each vertex $u \in V(Cayley(G_\alpha, S_j))$ $(1 \leq j \leq r)$ such that $u = (S_j \cdot 1) \cdot g$, where $g \in G_\alpha \setminus A$, we have

$$A_u = Lab(Cayley(G_\alpha, S_j), u) \cap A = g^{-1} S_j g \cap A = \{1\}.$$

Therefore $A(u) \simeq A$ and $|A(u)| = |A|$. Thus $V(Cayley(G_\alpha, S_j))$ form one A-orbit isomorphic to $A(v_j)$ of length n with $A_{S_j \cdot 1} = S_j = A_{v_j}$, and $c = (|V(C)| - n)/|A|$ A-orbits isomorphic to $A(u) \simeq A$ of length $|A|$ with, roughly speaking, a trivial A-stabilizer.

On the other hand, in $Cayley(G_\beta)$ the number of distinct A-orbits of length $|A|$ with the trivial A-stabilizer is $d = |V(Cayley(G_\beta)|/|A| = |G_\beta|/|A| = [G_\beta : A]$.

Let Γ_1 be the disjoint union of d isomorphic copies of Γ and cr isomorphic copies of $Cayley(G_\beta)$. Let Γ_2 be the union of disjoint unions of $m_j d$ isomorphic copies of $Cayley(G_\alpha, S_j)$, for all $1 \leq j \leq r$. Then both Γ_1 and Γ_2 have $m_j d$ distinct A-orbits of length n isomorphic to $A(v_j)$, and cdr different isomorphic A-orbits of length $|A|$.

Let $\{w_{ji} \mid 1 \leq i \leq m_j d,\ 1 \leq j \leq r\}$ and $\{u_{ji} \mid 1 \leq i \leq m_j d,\ 1 \leq j \leq r\}$ be the sets of representatives of the orbits of length n in Γ_1 and in Γ_2, respectively. Hence $A_{w_{ji}} = S_j = A_{u_{ji}}$, for all $1 \leq i \leq m_j d$ and $1 \leq j \leq r$.

Let $\{x_l \mid 1 \leq l \leq cdr\}$ and $\{y_l \mid 1 \leq l \leq cdr\}$ be the sets of representatives of the orbits of length $|A|$ in Γ_1 and in Γ_2, respectively. Then $A_{x_l} = \{1\} = A_{y_l}$, for all $1 \leq l \leq cdr$.

Let Γ' be the amalgam of Γ_1 and Γ_2 over these sets of vertices,

$$\Gamma' = \Gamma_1 *_{\{w_{ji} \cdot a = u_{ji} \cdot a \mid a \in A\} \cup \{x_l \cdot a = y_l \cdot a \mid a \in A\}} \Gamma_2.$$

By Corollary 8.4, Γ' is a finite precover of G such that the graphs Γ_1 and Γ_2 embed in it. Therefore the graph Γ embeds in Γ' as well. Moreover, by construction, the graph Γ' is $X_\beta^{\pm}$-saturated, and $n(\Gamma') = n(\Gamma) \setminus \{n\}$. Thus Γ' satisfies the inductive assumption. We are done.

◇

Lemma 8.7. *If A is cyclic then any finite $X_\beta^{\pm}$-saturated precover of G can be embedded in a cover of G with finitely many vertices ($\beta \in \{1, 2\}$).*

Proof. Since A is cyclic, $S_i = S_j$ for all $1 \leq i, j \leq m$, that is $A_{v_i} = A_{v_j}$. Assume that $A_v = S$, for all $v \in \{v_i \mid 1 \leq i \leq m\}$.

Consider $Cayley(G_\alpha, S)$ $(1 \leq \beta \neq \alpha \leq 2)$. For each vertex $u \in V(Cayley(G_\alpha, S))$, we have

$$A_u = Lab(Cayley(G_\alpha, S), u) \cap A = g^{-1}Sg \cap A,$$

where $g \in G_\alpha$, such that $(S \cdot 1) \cdot g = u$. Thus $|A_u| \leq |S|$ and therefore, since A is cyclic, $A_u \leq S \leq A$.

Claim 1. *Let* $\alpha \in \{1, 2\}$.

Then there is $0 < N \in \mathbf{Z}$ *such that* $Cayley(G_\alpha, S)$ *can be embedded into a finite* $X_\alpha^{\pm}$*-saturated precover* C *of* G*, whose* X_α*-monochromatic vertices form* N *distinct* A*-orbits of length* n *isomorphic to each other, with the* A*-stabilizer* S*. More precisely,*

$$VM_\alpha(C) = \bigcup_{i=1}^{N} A(v_i), \text{ such that } A_{v_i} = S \ (\forall\ 1 \leq i \leq N).$$

Proof of the Claim. The proof is by induction on the number of prime factors of $|S|$.

Assume first that $|S| = p$ is prime. By the above observation, for all $u \in V(Cayley(G_\alpha, S))$, either $A_u = S$, $|A(u)| = n$, or $A_u = \{1\}$, $|A(u)| = |A|$ (that is $A(u) \simeq A$).

Assume that $V(Cayley(G_\alpha, S))$ form b distinct isomorphic orbits of length n. Hence the number of distinct A-orbits of $V(Cayley(G_\alpha, S))$ of length $|A|$ isomorphic to $A(u)$ with the trivial A-stabilizer is $c = (|V(C)| - n \cdot b)/|A|$.

On the other hand, in $Cayley(G_\beta)$ the number of distinct A-orbits of length $|A|$ with the trivial A-stabilizer is

$$d = \frac{|V(Cayley(G_\beta))|}{|A|} = \frac{|G_\beta|}{|A|} = [G_\beta : A].$$

Let C_1 be the disjoint union of d isomorphic copies of $Cayley(G_\alpha, S)$. Let C_2 be the disjoint union of c isomorphic copies of $Cayley(G_\beta)$. Then both C_1 and C_2 have cd distinct A-orbits of length $|A|$.

Let $\{x_l \mid 1 \leq l \leq cd\}$ and $\{y_l \mid 1 \leq l \leq cd\}$ be the sets of representatives of these orbits in C_1 and in C_2, respectively. Then $A_{x_l} = \{1\} = A_{y_l}$, for all $1 \leq l \leq cd$.

Let C be the amalgam of C_1 and C_2 over these sets of vertices,

$$C = C_1 *_{\{x_l \cdot a = y_l \cdot a \ \mid \ a \in A\}} C_2.$$

By Corollary 8.4, C is a finite precover of G such that the graphs C_1 and C_2 embed in it. Therefore the graph $Cayley(G_\alpha, S)$ embeds in C as well. Moreover, by construction, the graph C is $X_\alpha^{\pm}$-saturated, and $VM_\alpha(C)$ form $N = bd$ distinct A-orbits of length n isomorphic to each other, with, roughly speaking, an A-stabilizer S.

Assume now that $|S|$ is not a prime number. Let $V(Cayley(G_\alpha, S))$ form t_i distinct A-orbits of length $\frac{|A|}{i}$ isomorphic to $A(u_i)$ with the A-stabilizer $A_{u_i} \leq S$. Thus $|A_{u_i}| = i$, where $i \in I = \{i \mid 1 \leq i < |S|,\ i \mid |S|\}$.

By the inductive assumption, $Cayley(G_\beta, A_{u_i})$ can be embedded into a finite $X_\beta^{\pm}$-saturated precover C_i of G whose X_β-monochromatic vertices form k_i distinct A-orbits isomorphic to $A(u_i)$ of length $\frac{|A|}{i}$ with the A-stabilizer A_{u_i}.

Let $l = lcm(\{k_i \mid i \in I\})$. We take C_1' be the disjoint union of l isomorphic copies of $Cayley(G_\alpha, S)$. Let C_2' be the union of disjoint unions of $\left(t_i \frac{l}{k_i}\right)$ isomorphic copies of C_i, for all $i \in I$. Then both C_1' and C_2' have $t_i l$ distinct A-orbits of length $|i|$ isomorphic to $A(u_i)$.

Let $\{w_{ij} \mid i \in I,\ 1 \leq j \leq t_i l\}$ and $\{u_{ij} \mid i \in I,\ 1 \leq j \leq t_i l\}$ be the sets of representatives of these orbits in C_1' and in C_2', respectively. Hence $A_{w_{ij}} = A_{u_i} = A_{u_{ij}}$, for all $i \in I$ and $1 \leq j \leq t_i l$.

Let C be the amalgam of C_1' and C_2' over these sets of vertices,

$$C = C_1' *_{\{w_{ij} \cdot a = u_{ij} \cdot a \mid a \in A\}} C_2'.$$

By Corollary 8.4, C is a finite precover of G such that the graphs C_1' and C_2' embed in it. Therefore the graph $Cayley(G_\alpha, S)$ embeds in C as well. Moreover, by construction, the graph C is $X_\alpha^{\pm}$-saturated, and $VM_\alpha(C)$ form $N = t_n l$ distinct A-orbits of length n isomorphic to $A(v)$ with the A-stabilizer S. We are done.

$\diamond$

Let Γ_1 be the disjoint union of N isomorphic copies of Γ and let Γ_2 be the disjoint unions of m isomorphic copies of C. Then both Γ_1 and Γ_2 have mN distinct A-orbits of length n isomorphic to $A(v)$ with the A-stabilizer S. The standard arguments used in the proofs of Lemmas 8.5 and 8.6 complete the proof.

$\diamond$

Proof of the Main Theorem. We first construct the graph $(\Gamma(H), v_0)$, using the generalized Stallings' folding algorithm.

Without loss of generality we can assume that g is a normal word. Since $g \notin H$, then, by Theorem 4.1 (3), $v_0 \cdot g \neq v_0$. Thus either g is readable in $\Gamma(H)$, that is $v_0 \cdot g = v \in V(\Gamma(H))$, or it is not readable.

Assume first that $v_0 \cdot g = v \in V(\Gamma(H))$. We apply the algorithm described along with the proof of Lemma 8.2 to embed the finite precover $Lab(\Gamma(H), v_0)$ into a finite $X_i^{\pm}$-saturated precover (Γ, ϑ), where ϑ is the image of v_0, and we take $1 \leq i \neq j \leq 2$, if A is malnormal or central in G_j.

Now we embed (Γ, ϑ) into a finite cover of G, using the appropriate algorithm given along with the proof of one of Lemmas 8.5, 8.6 or 8.7. Let (Φ, ν) be the resulting graph, where ν is the image of ϑ.

Let $K = Lab(\Phi, \nu)$. By Theorem 7.1, $[G : K] < \infty$ and $(\Phi, \nu) = (\Gamma(K), u_0)$. Since

$$\Gamma(H) \subseteq \Gamma \subseteq \Phi,$$

we have

$$Lab(\Gamma(H), v_0) \leq Lab(\Gamma, \vartheta) \leq Lab(\Phi, \nu).$$

Thus $H \leq K$. However $g \notin K$, because the above graphs are inclusions are embeddings. Therefore we are done.

Assume now that g is not readable in $\Gamma(H)$. Let g_1 be the longest prefix of g that is readable in $\Gamma(H)$, that is $v_0 \cdot g_1 = v \in V(\Gamma(H))$. Thus $v \in VM(\Gamma(H))$. Without loss of generality, we can assume that $v \in VM_1(\Gamma(H))$.

We glue to $\Gamma(H)$ a "stem" labelled by g_2 at v, where $g \equiv g_1 g_2$. Let Γ be the resulting graph (see Figure 6).

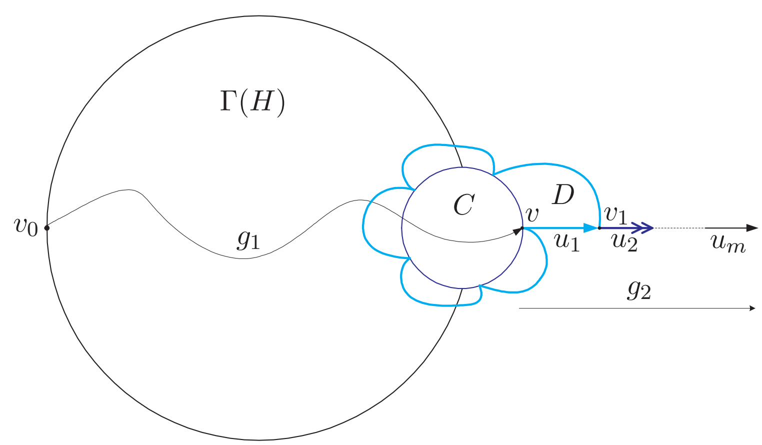

FIGURE 6.

Claim 2. *The graph (Γ, v_0) can be embedded into a finite precover (Γ', v'_0) of G such that $v'_0 \neq v'_0 \cdot g \in VM(\Gamma')$, where v'_0 is the image of v_0 in Γ'.*

Proof of the Claim. Let $(u_1, \cdots, u_m)$ be the normal (Serre) decomposition of g_2. Hence $u_1 \in G_2 \setminus A$. The proof is by induction on the syllable length of g_2.

Let C be a X_1-monochromatic component of $\Gamma(H)$ such that $v \in V(C)$. Let $S = A_v$.

Consider $Cayley(G_1, S, S \cdot 1)$. Thus $A_{S \cdot 1} = S$ and the A-orbit $A(S \cdot 1) = \{(S \cdot 1) \cdot x \mid x \in A\} = \{Sx \mid x \in A\} \subseteq V(Cayley(G_2, S))$ is isomorphic to the

A-orbit of v in C. Therefore taking $\Gamma_v = \Gamma *_{\{v \cdot x = Sx \mid x \in A\}} Cayley(G_2, S)$, we get a graph such that $\Gamma(H)$ and $Cayley(G_2, S)$ embed in it, by Corollary 8.4.

Let D be the X_2-monochromatic component of Γ_v such that $v \in V(D)$. Since $u_1 \in G_2$ and D is $X_2^{\pm}$-saturated, there exists a path γ in D such that $\iota(\gamma) = v$ and $lab(\gamma) \equiv u_1$. Moreover, the vertex $v_1 = \tau(\gamma) \in VB(D) \setminus VB(C)$, because $u_1 \in G_2 \setminus A$. Thus $v_1 \neq v_0$.

Therefore the graph Γ_v can be thought of as a precover of G with a stem labelled by $u_2 \cdots u_m$ which rises up from the vertex v_1. Note that $u_2 \in G_1 \setminus A$. Thus the graph Γ_v and the word given by the normal (Serre) decomposition $(u_2, \cdots, u_m)$ satisfy the inductive assumption. We are done.

$\diamond$

Proceeding in the same manner as in the previous case, when $v_0 \cdot g \in V(\Gamma(H))$, we embed the finite precover $Lab(\Gamma', v_0')$ of G into a finite cover (Φ, ν) of G. This completes the proof. $\diamond$

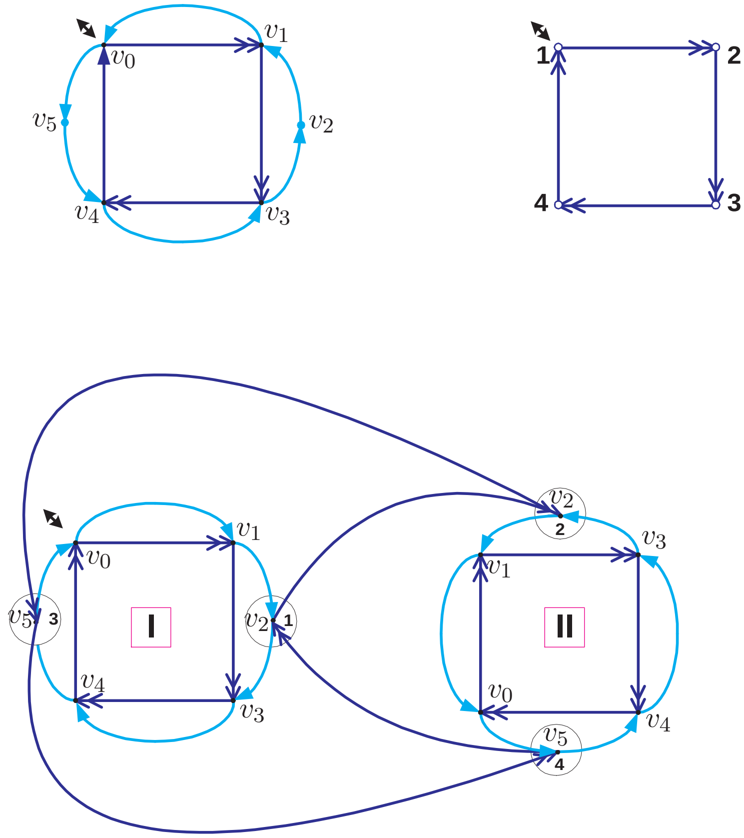

FIGURE 7. The construction of the cover $\Gamma(K)$ of G.

Example 8.8. Let G and H_1 be as in Example 6.9. Recall that

$$G = \langle x, y | x^4, y^6, x^2 = y^3 \rangle \text{ and } H_1 = \langle xy \rangle.$$

Let $g = xy^{-1}$ be an element of G. By Theorem 4.1 (3), $g \notin H_1$, because $v_0 \cdot g \neq v_0$.

Figure 7 illustrates the construction of the cover $\Gamma(K)$ of G, where $K \leq G$ is the separating subgroup for $H_1 \leq G$ and the element $g \notin H_1$. $\diamond$

Appendix A.

Below we follow the notation of Grunschlag [18], distinguishing between the "*input*" and the "*given data*", the information that can be used by the algorithm *"for free"*, that is it does not affect the complexity issues.

Algorithm

Given: Finite groups G_1, G_2, A and the amalgam $G = G_1 *_A G_2$ given via (1.a), (1.b) and (1.c), respectively.

We assume that the Cayley graphs and all the relative Cayley graphs of the free factors are given.

Input: A finite set $\{g_1, \cdots, g_n\} \subseteq G$.

Output: A finite graph $\Gamma(H)$ with a basepoint v_0 which is a reduced precover of G and the following holds

- $Lab(\Gamma(H), v_0) =_G H$;
- $H = \langle g_1, \cdots, g_n \rangle$;
- a normal word w is in H if and only if there is a loop (at v_0) in $\Gamma(H)$ labelled by the word w.

Notation: Γ_i is the graph obtained after the execution of the i-th step.

Step1: Construct a based set of n loops around a common distinguished vertex v_0, each labelled by a generator of H;

Step2: Iteratively fold edges and cut hairs [6];

Step3:

`For` each X_i-monochromatic component C of Γ_2 $(i = 1, 2)$ `Do`

`Begin`

pick an edge $e \in E(C)$;

glue a copy of $Cayley(G_i)$ on e via identifying 1_{G_i} with $\iota(e)$

and identifying the two copies of e in $Cayley(G_i)$ and in Γ_2;

`If` necessary `Then` iteratively fold edges;

`End;`

Step4:

`For` each $v \in VB(\Gamma_3)$ `Do`

[6] A *hair* is an edge one of whose endpoint has degree 1

`If` there are paths p_1 and p_2, with $\iota(p_1) = \iota(p_2) = v$ and $\tau(p_1) \neq \tau(p_2)$ such that

$$lab(p_i) \in G_i \cap A \ (i = 1, 2) \text{ and } lab(p_1) =_G lab(p_2)$$

`Then` identify $\tau(p_1)$ with $\tau(p_2)$;
`If` necessary `Then` iteratively fold edges;

Step5: Reduce Γ_4 by an iterative removal of all (*redundant*) X_i-monochromatic components C such that

- (C, ϑ) is isomorphic to $Cayley(G_i, K, K \cdot 1)$, where $K \leq A$ and $\vartheta \in VB(C)$;
- $|VB(C)| = [A : K]$;
- one of the following holds
  - $K = \{1\}$ and $v_0 \notin VM_i(C)$;
  - K is a nontrivial subgroup of A and $v_0 \notin V(C)$.

Let Γ be the resulting graph;

`If` $VB(\Gamma) = \emptyset$ and (Γ, v_0) is isomorphic to $Cayley(G_i, 1_{G_i})$
`Then` we set $V(\Gamma_5) = \{v_0\}$ and $E(\Gamma_5) = \emptyset$;
`Else` we set $\Gamma_5 = \Gamma$.

Step6:
`If`

- $v_0 \in VM_i(\Gamma_5)$ $(i \in \{1, 2\})$;
- (C, v_0) is isomorphic to $Cayley(G_i, K, K \cdot 1)$, where $L = K \cap A$ is a nontrivial subgroup of A and C is a X_i-monochromatic component of Γ_5 such that $v_0 \in V(C)$;

`Then` glue to Γ_5 a X_j-monochromatic component $(1 \leq i \neq j \leq 2)$ $D = Cayley(G_j, L, L \cdot 1)$ via identifying $L \cdot 1$ with v_0 and identifying the vertices $L \cdot a$ of $Cayley(G_j, L, L \cdot 1)$ with the vertices $v_0 \cdot a$ of C, for all $a \in A \setminus L$.

Denote $\Gamma(H) = \Gamma_6$.

Remark A.1. Note that the first two steps of the above algorithm correspond precisely to the Stallings' folding algorithm for finitely generated subgroups of free groups [53, 37, 25]. $\diamond$

Example A.2. Let $G = gp\langle x, y | x^4, y^6, x^2 = y^3 \rangle$.

Let H_1 and H_2 be finitely generated subgroups of G such that

$$H_1 = \langle xy \rangle \text{ and } H_2 = \langle xy^2, yxyx \rangle.$$

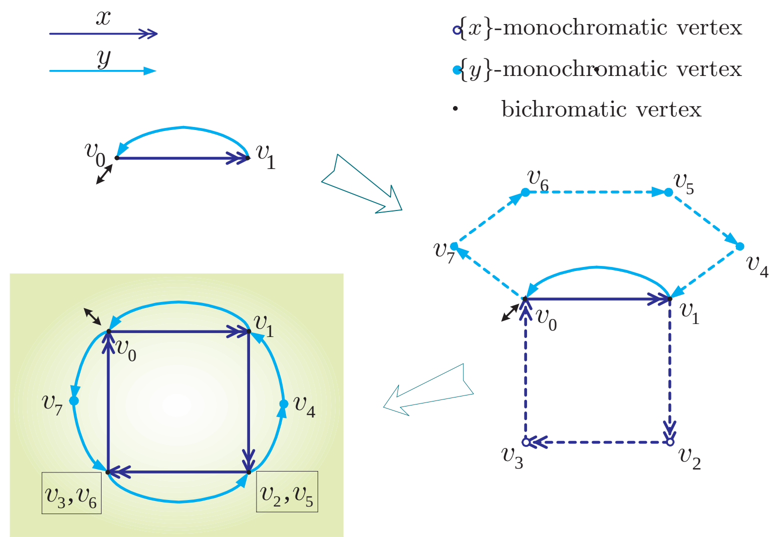

FIGURE 8. The construction of $\Gamma(H_1)$.

The construction of $\Gamma(H_1)$ and $\Gamma(H_2)$ by the algorithm presented above is illustrated on Figures 8 and 9. $\diamond$

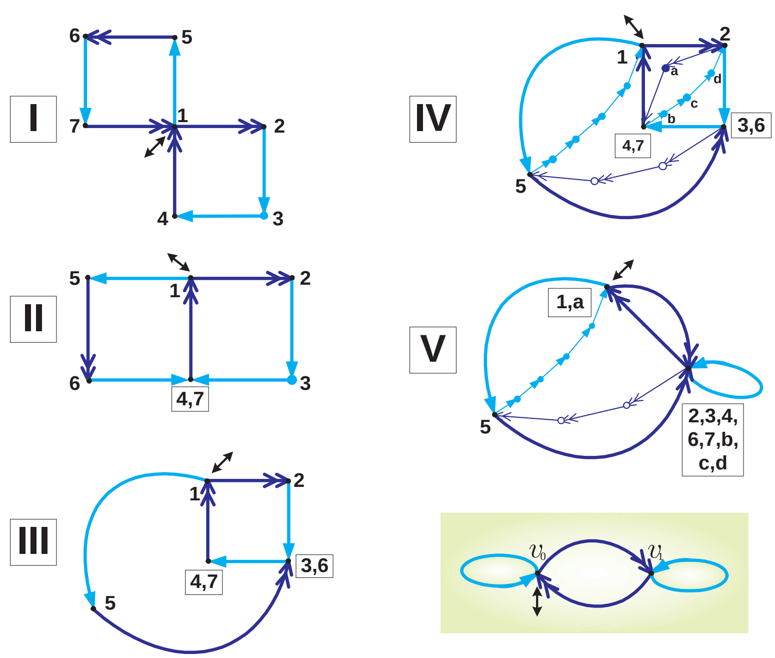

FIGURE 9. The construction of $\Gamma(H_2)$.

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